

The spectral theorem for a compact, self-adjoint operator.

([R, Thm 16.6, Cor. 16.7] for $\mathcal{H}$ separable; [B, Theorem 4.3.4 (2), (e), (f)] for $\mathcal{H}$ arbitrary.)

Let $T \in \mathcal{K}(\mathcal{H})$ such that $T \neq 0$ and $T = T^*$. Let $E_\lambda = \ker(T - \lambda I)$, $\lambda \neq 0$.
Let $L = \{\lambda \in \sigma_p(T) \mid \lambda \neq 0\}$. Let ε_λ be an o.n. basis for E_λ , and set

$$(I) \quad \varepsilon' = \bigcup_{\lambda \in L} \varepsilon_\lambda.$$

Then ε' is a ^{countable} orthonormal basis for $\overline{T(\mathcal{H})}$. Moreover, $\overline{T(\mathcal{H})} = (\ker T)^\perp$ (So $\mathcal{H} = \ker T \oplus \overline{T(\mathcal{H})}$).

(II) If $\ker T = \{0\}$, set $\varepsilon_0 := \emptyset$, otherwise let ε_0 be an o.n.b. for $\ker(T)$. Then $\varepsilon := \varepsilon_0 \cup \varepsilon'$ is an o.n.b. for $\mathcal{H}$ consisting of

ε_0 may not be countable if $\mathcal{H}$ is not separable)

eigenvectors corresponding to $\lambda = 0$

eigenvectors corresponding to $\lambda \in \sigma_p(T) \setminus \{0\}$

eigenvectors for T .

III Let $P_\lambda = P_{E_\lambda}$ for each $\lambda \in \sigma_p(T)$, $\lambda \neq 0$. Then, $P_\lambda P_{\lambda'} = 0$ when

$$\lambda \neq \lambda' \text{ in } \sigma_p(T) \setminus \{0\}. \quad \left[\text{For every } x \in \mathcal{H}, Tx = \sum_{\lambda \in L} \lambda P_\lambda(x). \quad (1) \right]$$

Moreover, (2) $T = \sum_{\lambda \in L} \lambda P_\lambda$ in $\mathcal{B}(\mathcal{H})$ (as convergence of a series of operators.)

Note: If $\mathcal{H}$ is separable, then write $\varepsilon_0 = \{u_j^0\}_{j \in \mathbb{N}}$, $\varepsilon' = \{u_j^1\}_{j \in \mathbb{N}}$ and $\varepsilon = \{e_j\}_{j \in \mathbb{N}} = \{u_j^0\}_{j \in \mathbb{N}} \cup \{u_j^1\}_{j \in \mathbb{N}}$. Then identity (1) becomes:

$$(1') \quad Tx = \sum_{j=1}^{\infty} \lambda_j \langle x, w_j \rangle w_j \text{ for all } x \in \mathcal{H}. \quad [\text{R, Corollary 16.7}]$$

Note: [R, Theorem 16.6] is (1) in part (III) above ([B, Thm 4.3.4 (f)]).

Indeed, note that if $\ker T \neq 0$, so that $0 \in \sigma_p(T)$, an eigenvector u corresponding to $\lambda = 0$ will not appear in the sum.

To prove the Spectral Theorem, we need:

Lemma ([R, Lemma 16.5])

Let $T \in B(\mathcal{H})$ and $Y \subseteq \mathcal{H}$ a closed subspace s.t. $TY \subseteq Y$ (this says that Y is an invariant subspace for T). Then $Y^\perp$ is an invariant subspace for T^* . Thus, if $T = T^*$, we have:

$$TY \subseteq Y \Rightarrow TY^\perp \subseteq Y^\perp$$

Proof: Assume $TY \subseteq Y$. For $x \in \mathcal{H}$, $y \in Y$, $\langle x, Ty \rangle = \langle T^*x, y \rangle$ shows that $T^*x \in Y^\perp$ if $x \in Y^\perp$.

Proof of the spectral theorem. Assume $\sigma_p(T) \setminus \{0\}$ is infinite. So, it is countable.

(I) Given $\lambda \in \sigma_p(T) \setminus \{0\}$, we know that $E_\lambda = \ker(T - \lambda I)$ is finite dimensional. We may choose a (finite) orthonormal basis E_λ for E_λ .

Given $\lambda \neq \lambda'$ in $\sigma_p(T) \setminus \{0\}$ we have seen: $E_\lambda \perp E_{\lambda'}$. Thus,

$\{E_\lambda \cup E_{\lambda'}\}$ form an orthonormal (finite set) for $\lambda \neq \lambda'$. In fact,

"We get that $\mathcal{E}' := \bigcup_{\lambda \in \sigma_p(T) \setminus \{0\}} E_\lambda$ is an orthonormal

family which is countable as a countable union of finite sets.

Let $M = \overline{\text{Span}(\mathcal{E})}$. Will show $M = \overline{T(\mathcal{H})}$.

Know $\overline{T(\mathcal{H})} = (\overline{T(\mathcal{H})}^\perp)^\perp \stackrel{\text{Prop. 2.4.10}}{=} (\ker T^*)^\perp \stackrel{T=T^*}{=} (\ker T)^\perp$. } Need $M = (\ker T)^\perp$.

Since $\ker T$ is closed, $M = (\ker T)^\perp \iff M^\perp = \ker T$.

To finish the proof of (I) it suffices to prove that

$$M^\perp = \ker T.$$

" $\supseteq$ " Let $x \in \ker T$, Need $\langle x, y \rangle = 0$ for every $y \in M$. 30/09/2025 ③

Since $M = \overline{\text{Span}(\mathcal{E}')}$, it suffices to take y of the form $y = u_\lambda$, where $u_\lambda \in E_\lambda$ for some $\lambda \in \sigma_p(T) \setminus \{0\}$. Then $Ty = \lambda y$.

$$\Rightarrow \langle x, y \rangle = \langle x, \lambda y \rangle = \langle x, Ty \rangle = \langle Tx, y \rangle = \langle 0, y \rangle = 0$$

Since $\lambda \neq 0$, get $x \in M^\perp$, as needed. $T = T^*$

" $\subseteq$ " To show $M^\perp \subseteq \ker T$, we assume for a contradiction

that $\exists v \in M^\perp$ such that $Tv \neq 0$. We form the restriction

operator $T|_{M^\perp}: M^\perp \rightarrow H$, $T|_{M^\perp}(v) = Tv$ for $v \in M^\perp$.

Since $M = \overline{\text{Span}(\mathcal{E}')}$, we get $T(M) \subseteq M$. (If $y \in T(M)$, then y is a limit of $T(y_n)$, $y_n \in \text{Span}(\mathcal{E}')$, but $T(\text{Span} \mathcal{E}') \subseteq \text{Span} \mathcal{E}'$).

By the previous lemma, also $T(M^\perp) \subseteq M^\perp$. We obtain a non-zero self-adjoint operator $T|_{M^\perp}: M^\perp \rightarrow M^\perp$ which is still compact (as a restriction of T). By the key result, it admits a non-zero eigenvalue, call it λ_0 , with $\lambda_0 \in \mathbb{R}$. Let $v_0 \in M^\perp$ be an eigenvector corresponding to λ_0 , thus $v_0 \neq 0$.

Since $\ker[(T - \lambda_0 I)|_{M^\perp}] \neq 0$, also $\ker(T - \lambda_0 I) \neq 0$, so $\lambda_0 \in \sigma_p(T) \setminus \{0\}$. Then $v_0 \in \mathcal{E}' \subseteq M$, so $v_0 \in M \cap M^\perp = \{0\}$, a contradiction.

We conclude $M^\perp = \ker(T)$, which finishes the proof of (I).

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⑦

(II) From (I), we have $\mathcal{H} = \ker T \oplus \overline{T(\mathcal{H})}$.

If $\ker T = \{0\}$, then $\mathcal{E}' = \bigcup_{\lambda \in \sigma_p(T) \setminus \{0\}} \mathcal{E}_\lambda$ is an o.n.b for $\mathcal{H}$.

Assume $\ker T \neq \{0\}$ and let $\mathcal{E}_0 = \{u_\lambda^0\}_{\lambda \in \mathcal{N}}$ be an o.n.b for $\ker T$.

Then $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}'$ is an orthonormal set.

We have $\overline{\text{Span}(\mathcal{E})} = \mathcal{H}$, because every $x \in \mathcal{H}$ decomposes as $x = x_{M^\perp} + x_M$ with $x_{M^\perp} \in \overline{\text{Span} \mathcal{E}_0}$ and $x_M \in \overline{\text{Span}(\mathcal{E}')} = M$.

(III) General observation

If M_1, M_2 are closed subspaces of $\mathcal{H}$ such that $M_1 \perp M_2$, then $P_{M_1} \cdot P_{M_2} = 0$. (Easy verification).

Therefore, $P_\lambda P_{\lambda'} = 0$ for all $\lambda \neq \lambda'$ in $\sigma_p(T) \setminus \{0\}$.

Proof of (1): Write $\sigma_p(T) \setminus \{0\}$ as $\{\lambda_k\}_{k \in \mathbb{N}}$, $\lambda_k \in \mathbb{R}$, $\forall k \geq 1$. Choose

$\mathcal{E}_{\lambda_k} := \{v_{k,1}, \dots, v_{k,n_k}\}$ o.n.b for E_{λ_k} , for $k \geq 1$.

Then, $P_{\lambda_k}(x) = \sum_{l=1}^{n_k} \langle x, v_{k,l} \rangle v_{k,l}$ for every k . Related as P_k .

Have $\mathcal{E}' = \{v_{1,1}, \dots, v_{1,n_1}, v_{2,1}, \dots, v_{2,n_2}, \dots, v_{k,1}, \dots, v_{k,n_k}, \dots\}$ o.n.b. for M .

Think of it: $u_1, u_2, \dots$

Given $x \in \mathcal{H}$, $Tx \in \overline{T(\mathcal{H})}$, so $Tx = \sum_{k=1}^{\infty} \langle Tx, u_k \rangle u_k = \lim_{M \rightarrow \infty} \sum_{k=1}^M \langle Tx, u_k \rangle u_k$

$\forall k \geq 1$, blocks of n_k

$= \lim_{M \rightarrow \infty} \sum_{k=1}^M \sum_{l=1}^{n_k} \langle Tx, v_{k,l} \rangle v_{k,l}$, use $\langle Tx, v_{k,l} \rangle = \lambda_k \langle x, v_{k,l} \rangle$

$= \lim_{M \rightarrow \infty} \sum_{k=1}^M \lambda_k \sum_{l=1}^{n_k} \langle x, v_{k,l} \rangle v_{k,l} = \lim_{M \rightarrow \infty} \sum_{k=1}^M \lambda_k P_{\lambda_k}(x)$, as claimed!

To finish the proof of the Spectral Theorem, we show that in $B(H)$, we have: (2) $T = \sum_{\lambda \in L} \lambda P_{\lambda}$, where $L = \{\lambda \in \sigma_p(T) \mid \lambda \neq 0\}$.

Case: L finite. This happens when $T \in F(H)$. Equation (2) is a routine verification.

Case: L is countably infinite. Write it as $L = \{\lambda_k\}_{k \in \mathbb{N}}$. Then (2) means

$$\lim_{n \rightarrow \infty} \|T - \sum_{k=1}^n \lambda_k P_{\lambda_k}\| = 0.$$

We know that $\lim_{k \rightarrow \infty} \lambda_k = 0$. Let $\varepsilon > 0$. There exists $K \in \mathbb{N}$ s.t.

$$k \geq K \Rightarrow |\lambda_k| < \varepsilon.$$

Take $x \in H$. Then $\|T - \sum_{k=1}^n \lambda_k P_{\lambda_k}\| x\|^2 = \|Tx - \sum_{k=1}^n \lambda_k P_{\lambda_k} x\|^2$

$$\stackrel{(1)}{=} \left\| \sum_{k=n+1}^{\infty} \lambda_k P_{\lambda_k} x \right\|^2 = \sum_{k=n+1}^{\infty} |\lambda_k|^2 \|P_{\lambda_k}(x)\|^2 \quad (\text{Pythagoras})$$

$$\text{for } n > K \rightarrow < \varepsilon^2 \sum_{k=n+1}^{\infty} \left(\sum_{l=1}^{n_k} |\langle x, u_{k,l} \rangle|^2 \right) < \varepsilon^2 \|x\|^2.$$

We get $\sup_{\|x\| \leq 1} \|T - \sum_{k=1}^n \lambda_k P_{\lambda_k}\| < \varepsilon$ as $n > K$. This proves (2). $\square$

Note: There is an often used alternative for writing (1):

List the eigenvalues repeated as many times as their multiplicity, so

$$\{\underbrace{\lambda_1, \dots, \lambda_1}_{n_1}, \underbrace{\lambda_2, \dots, \lambda_2}_{n_2}, \dots, \underbrace{\lambda_k, \dots, \lambda_k}_{n_k}, \dots\} = \{s_j\}_{j \in \mathbb{N}} \text{ and}$$

list the eigenvectors as $\{u_1, u_2, u_3, \dots\}$. Then: $\forall x \in H$,

$$Tx = \sum_{j=1}^{\infty} s_j \langle x, u_j \rangle u_j. \quad (\text{See [R, equation (16.5)]})$$