

Theorem [R, Theorem 16.8] Let $T \in K(H)$, $T = T^*$, Hermitian. Then either $\sigma(T) = \sigma_p(T)$ or $\sigma(T) = \sigma_p(T) \cup \{0\}$ and

either $\sigma(T) = \{0\}$ or $\sigma(T) \neq \{0\}$

W. Proof: Assume H is infinite dimensional. We show: $\sigma(T) \subseteq \overline{\sigma_p(T)}$.
 By the Spectral Theorem, there exists an o.n.b. $\{e_j\}_{j \in \mathbb{N}}$ for consisting of eigenvectors for T such that for every $x \in H$ we have:

$$Tx = \sum_{j=1}^{\infty} \lambda_j \langle x, e_j \rangle e_j, \text{ where } \lambda_j \text{ is eigenvalue corresponding to } e_j.$$

(Note: $\lambda_j \neq 0$ only for those e_j in $M = \overline{\text{span}\{e_j\}}$)

Take $\lambda \notin \overline{\sigma_p(T)}$. Then $\exists \delta > 0$ s.t. $\sup_{j \in \mathbb{N}} |\lambda - \lambda_j| \geq \delta > 0$. (Here we use that $\lambda \neq 0$)

Claim: $T - \lambda I$ is invertible (which will imply $\lambda \notin \sigma(T)$).

Note: $(T - \lambda I)x = y \Leftrightarrow \sum_{j=1}^{\infty} (\lambda_j - \lambda) \langle x, e_j \rangle e_j = y = \sum_{j=1}^{\infty} \langle y, e_j \rangle e_j$.

For each $i \in \mathbb{N}$, take inner prod with e_i in each term on both sides,

to get $(T - \lambda I)x = y \Leftrightarrow (\lambda_i - \lambda) \langle x, e_i \rangle \stackrel{(*)}{=} \langle y, e_i \rangle$ for every $i \in \mathbb{N}$.

Now, $\lambda_i \neq \lambda$ for every $i \in \mathbb{N}$.

(*) has solution $x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$

with $\langle x, e_i \rangle = \frac{1}{\lambda_i - \lambda} \langle y, e_i \rangle$ for every i .

Note: $\|x\|^2 = \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 = \sum_{i=1}^{\infty} \frac{1}{|\lambda_i - \lambda|^2} |\langle y, e_i \rangle|^2 \leq \frac{1}{\delta^2} \sum_{i=1}^{\infty} |\langle y, e_i \rangle|^2 = \frac{\|y\|^2}{\delta^2}$

So we can define $(T - \lambda I)^{-1}: H \rightarrow H$ by $y \mapsto \sum_{i=1}^{\infty} \frac{1}{\lambda_i - \lambda} \langle y, e_i \rangle e_i$

Fact $\sigma(T)$ is closed (Will not prove this here).

Since $\sigma_p(T) \subseteq \sigma(T)$ by def, also $\overline{\sigma_p(T)} \subseteq \overline{\sigma(T)}$ } So $\overline{\sigma_p(T)} \subseteq \sigma(T)$

In all, $\overline{\sigma_p(T)} = \overline{\sigma(T)} = \sigma(T)$.

By the key result, the only possible limit point of $\sigma_p(T)$ is 0.

Let $\lambda \in \sigma(T)$. Then, $\lambda \in \overline{\sigma_p(T)}$, so $\lambda \in \sigma_p(T)$ or $\lambda = 0$.

This shows: every non-zero $\lambda \in \sigma(T)$ is an eigenvalue.

For ∞ -dim, $0 \in \sigma(T)$ by earlier. Also, 0 only possible limit point

of $\sigma_p(T)$, so: either $\sigma(T) = \sigma_p(T)$ (if 0 is an eigenvalue)
or $\sigma(T) = \sigma_p(T) \cup \{0\}$ (if 0 not an eigenvalue)

More general form: Given T as in theorem, for any $\lambda \in \mathbb{C} \setminus \{0\}$,

one of the following alternatives holds for the equation

$$(T - \lambda I)x = y \quad (*)$$

(1) $\lambda \notin \sigma_p(T)$ or (2) $\lambda \in \sigma_p(T)$.

In case (1), (*) has a unique solution $x = \sum_{j=1}^{\infty} \frac{\langle y, e_j \rangle}{\lambda_j - \lambda} e_j$.

In case (2), (*) has a solution if and only if $\langle y, e_j \rangle = 0$ for each j s.t. $\lambda = \lambda_j$. The set of solutions to (*) has form

$$x = \sum_{j=1}^{\infty} \frac{\langle y, e_j \rangle}{\lambda_j - \lambda} e_j + z, \text{ where } z \in \ker(T - \lambda I).$$

Lecture 14

Sturm-Liouville Problems: an application of the Spectral Theorem for compact self-adjoint operators

Motivation: Solve a second order linear differential equation of the form

$$(S-Lpr.) \quad -(p(x)u')' + q(x)u = \lambda u, \quad \text{with } u(a) = u(b) = 0,$$

where $p \in C^1[a, b] = \{f: [a, b] \rightarrow \mathbb{R} \mid f \text{ continuous \& differentiable, with } f' \text{ continuous}\}$,

$$q \in C^2[a, b] = \{f: [a, b] \rightarrow \mathbb{R} \mid f, f', f'' \text{ defined and continuous}\},$$

under suitable assumptions. Here $[a, b] \subseteq \mathbb{R}$ is a closed interval.

Assumptions (See [R, Chp. 17])

(i) $p(x) > 0$ on $[a, b]$.

(ii) $q(x) \geq 0$ on $[a, b]$.

What do we mean by "solve (S-Lpr)?" It means finding $\lambda \in \mathbb{R}$ for which there exists $u \in C^2[a, b]$ s.t. (S-Lpr) holds.

Say: λ is an eigenvalue for the Sturm-Liouville problem, and u is the corresponding eigenfunction.

Notation: $L[u] = -(p(x)u')' + q(x)u$.

$$\mathcal{D} := \{u \in C^2[a, b] \mid u(a) = u(b) = 0\}, \text{ the domain of } L.$$

Will show: The eigenvectors of the Sturm-Liouville problem form an o.n.b. for $L^2(a, b)$.

Essentially, the point is to construct an integral operator T so that $T \in \mathcal{K}(L^2(a, b))$, $T = T^*$, all eigenvalues of T are real, positive and simple (have multiplicity one). T is an "inverse" for L .

Sturm-Liouville problems

1) Symmetry of L and the Wronskian (the Wronski determinant)

Warning: L itself is not a bounded operator on a Hilbert space.

It is not defined on the entire $L^2(a, b)$, since $Du = u'$ is not bounded. (Take for example $f_n(x) = \sin(nx)$, $f_n \in C[a, b]$, $\|f_n\|_{\infty} = 1$, $\forall n$, $(Df_n)(x) = (\sin(nx))' = n \cos(nx)$, so $\|Df_n\| = n$. Thus $\sup_n \|Df_n\| \rightarrow \infty$, so D cannot be bounded.)

Lemma $L: D \rightarrow L^2(a, b)$ is symmetric, meaning that for $u, v \in D$,

$$\langle L[u], v \rangle = \langle u, L[v] \rangle$$

Further, for all $u \in D$ we have $\langle L[u], u \rangle \geq 0$ with $\langle L[u], u \rangle = 0$ if and only if $u = 0$.

Proof Given $u, v \in D$, we compute

$$\langle L[u], v \rangle = \int_a^b \left[(-p(x)u'(x))' + q(x)u(x) \right] v(x) dx$$

$$= \left[-p(x)u'(x)v(x) \right]_a^b + \int_a^b p(x)u'(x)v'(x) dx + \int_a^b q(x)u(x)v(x) dx$$

$$= - \int_a^b (p(x)v'(x))' u(x) dx + \left[p(x)v'(x)u(x) \right]_a^b + \int_a^b q(x)v(x)u(x) dx$$

$$= \langle u, L[v] \rangle. \text{ For } u=v, \langle L[u], u \rangle = \int_a^b p(x) [u'(x)]^2 dx + \int_a^b q(x) u(x)^2 dx,$$

thus $\langle L[u], u \rangle \geq 0$. Further, $\langle L[u], u \rangle = 0$, we must have $u'(x) = 0$ on $[a, b]$, so $u(x) = C$ for some C , $\forall x \in [a, b]$. But $u(a) = 0$, so $C = 0$ showing $u = 0$. $\square$

Corollary Suppose that λ is an eigenvalue of the (S-L_{pr}):

$$-(p(x)u')' + q(x)u = \lambda u, \quad u(a) = u(b) = 0.$$

Then $\lambda > 0$. Further, if u_1 and u_2 in $\mathcal{D}$ are eigenfunctions corresponding to distinct eigenvalues λ_1 and λ_2 , then $\langle u_1, u_2 \rangle = 0$.

Proof: If $L[u] = \lambda u$ with $\lambda \neq 0$, then

$$\lambda \|u\|^2 = \lambda \langle u, u \rangle = \langle L[u], u \rangle > 0 \text{ by Lemma. Thus, } \lambda > 0.$$

Assume $L[u_1] = \lambda_1 u_1$, $L[u_2] = \lambda_2 u_2$ for $\lambda_1 \neq \lambda_2$. Then

$$\lambda_1 \langle u_1, u_2 \rangle = \langle L[u_1], u_2 \rangle \stackrel{\text{Lemma}}{=} \langle u_1, L[u_2] \rangle = \lambda_2 \langle u_1, u_2 \rangle,$$

so $\langle u_1, u_2 \rangle = 0$.

Lemma Suppose $u_1, u_2 \in C^2[a, b]$ are nonzero solutions of $-(p(x)u')' + w(x)u = 0$, where $p \in C^1[a, b]$ satisfies $p(x) > 0$ on $[a, b]$ and $w \in C[a, b]$. Define the Wronskian (Wronski determinant):

$$W_p(u_1, u_2)(x) := p(x) [u_1'(x) u_2(x) - u_2'(x) u_1(x)],$$

Then W_p is constant. Moreover, $W_p(u_1, u_2) \neq 0$ if and only if u_1 and u_2 are linearly independent.

(This result is stated for arbitrary w , so not necessarily $w(x) \geq 0$, and u_1, u_2 are not required to be in $\mathcal{D}$).

Proof: Compute $\frac{d}{dx} W_p$ and use that $L[u_1] = 0 = L[u_2]$.

$$\text{Thus, we know } -[p'(x)u_i'(x) + p(x)u_i''(x)] + w(x)u_i(x) = 0, \quad i=1,2.$$

$$\text{So } pu_i'' = Wu_i - p'u_i' \text{ for } i=1,2.$$

01/10/2025 (6)

We get:

$$W_p' = p' [u_1' u_2 - u_2' u_1] + p [u_1'' u_2 + u_1' u_2' - u_2'' u_1 - u_2' u_1']$$

cancel out

$$= p' u_1' u_2 - p' u_2' u_1 + (w u_1 - p' u_1') u_2 - (w u_2 - p' u_2') u_1$$

cancel

$$= w u_1 u_2 - w u_2 u_1 = 0.$$

Therefore $W_p(x) = \text{constant}$ on $[a, b]$. Using that $p(x) > 0$ on $[a, b]$, we see that $W_p(u_1, u_2) \equiv 0 \iff u_1'(x)u_2(x) - u_2'(x)u_1(x) = 0, \forall x \in [a, b]$

Suppose $\alpha u_1(x) + \beta u_2(x) = 0$ for every $x \in [a, b]$, with $\alpha, \beta \in \mathbb{R}$.

Assuming $u_1'(x)u_2(x) - u_2'(x)u_1(x) \neq 0$ at all $x \in [a, b]$, we must show that $\alpha = \beta = 0$.

Differentiation gives $\alpha u_1'(x) + \beta u_2'(x) = 0$ for every $x \in [a, b]$.

Consider $\begin{cases} \alpha u_1(x) + \beta u_2(x) = 0 \\ \alpha u_1'(x) + \beta u_2'(x) = 0 \end{cases}$, with matrix $\begin{bmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{bmatrix}$. Since

This has determinant $u_1(x)u_2'(x) - u_2'(x)u_1(x) \neq 0$ at every $x \in [a, b]$,
the system has only the trivial solution $\alpha = \beta = 0$.

In case the determinant is zero at some (hence every) $x \in [a, b]$, then there are non-zero solutions $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so u_1 & u_2 are lin. dependent.

Corollary If λ is an eigenvalue of $(S-L_{\text{pr}})$, then $E_\lambda = \{u \in D \mid L[u] = \lambda u\}$ is one-dimensional.

Proof: If $u_1, u_2 \in E_\lambda$ then u_1, u_2 solve $-(p(x)u')' + (q - \lambda)u = 0$, hence $W_p(u_1, u_2) = \text{constant} = 0$ from $u_2(a) = u_1(a) = 0$. By lemma, u_1 & u_2 are lin. dependent. $\square$

The Green's function

Theorem Suppose that u_1, u_2 are lin. independent solutions in D of $-\frac{d}{dx}(p(x)\frac{dy}{dx}) + q(x)y = 0$ with boundary conditions $u_1(a) = 0$ and $u_2(b) = 0$. Set $W = W_p(u_1, u_2)$ and define

$G: [a, b] \times [a, b] \rightarrow \mathbb{R}$ by

$$G(x, y) = \begin{cases} W^{-1}u_1(x)u_2(y) & \text{if } a \leq x < y \\ W^{-1}u_2(x)u_1(y) & \text{if } y \leq x \leq b. \end{cases}$$

For $f \in C([a, b])$, define $u(x) = \int_a^b G(x, y) f(y) dy$. Then $u \in D$ and $L[u] = f$.

Proof: Need $u \in C^2([a, b])$. Note, we will have $u(a) = 0$ since $G(a, y) = 0$ and $u(b) = 0$ since $G(b, y) = 0$. Split the integral as $\int_a^x + \int_x^b$

u1: $a \leq y < x$ (and swap x & y in def. of G) $x \leq y < b$

$$\text{Have } u(x) = W^{-1} \int_a^x u_1(y)u_2(x)f(y)dy + W^{-1} \int_x^b u_2(y)u_1(x)f(y)dy$$
$$= \frac{u_2(x)}{W} \int_a^x u_1(y)f(y)dy + \frac{u_1(x)}{W} \int_x^b u_2(y)f(y)dy. \text{ Now differentiate.}$$

$$\Rightarrow u'(x) = \frac{u_2'(x)}{W} \int_a^x u_1(y)f(y)dy + \frac{u_2(x)}{W} u_1(x)f(x) + \frac{u_1'(x)}{W} \int_x^b u_2(y)f(y)dy - \frac{u_1(x)}{W} u_2(x)f(x)$$

↑
cancel out

$$\text{So } u'(x) = \frac{u_2'(x)}{W} \int_a^x u_1(y)f(y)dy + \frac{u_1'(x)}{W} \int_x^b u_2(y)f(y)dy.$$

Compute u'' , using that $W = p(u_1' u_2 - u_2' u_1)$. Get u'' cont & $L[u] = f$.