

Sturm-Liouville theory

Recall: Solve $[L[u] = \lambda u]$, where $L: \mathcal{D} \subset L^2[a, b] \rightarrow L^2[a, b]$,

$\mathcal{D} = \{u \in C^2[a, b] \mid u(a) = u(b) = 0\}$, $[L[u] := -(p(x)u')' + q(x)u]$ for $p \in C^1[a, b]$, $p(x) \geq 0$ for every $x \in [a, b]$ and $q \in C^2[a, b]$, $q(x) \geq 0, \forall x$.

Fact: $\mathcal{D}$ is not closed in $\|\cdot\|_2$ -norm, so it is not a Hilbert space.

Have seen: (i) L symmetric, ie. $\langle L[u], v \rangle = \langle u, L[v] \rangle, \forall u, v \in \mathcal{D}$.

(ii) Any eigenvalue λ of the Sturm-Liouville problem is positive (> 0).

(iii) $\langle L[u], u \rangle = 0 \Leftrightarrow u = 0$.

(iv) For $u_1, u_2 \in C^2[a, b]$ solving $[-(p(x)u')' + w(x)u = 0]$ for p given as above and $w \in C[a, b]$, the Wronskian

$W_p(u_1, u_2)(x) := p(x) [u_1'(x)u_2(x) - u_2'(x)u_1(x)]$ is constant on $[a, b]$ and is non-zero precisely when (iff) u_1, u_2 are linearly independent. Set $W := W_p(u_1, u_2)$.

The Green's function ([R, Thm 17.5])

for $w(x) = q(x) \geq 0, \forall x$.

Suppose u_1, u_2 in $C^2[a, b]$ are lin. indep. solutions of (2) with $u_1(a) = 0$ and $u_2(b) = 0$. The function $G: [a, b] \times [a, b] \rightarrow \mathbb{R}$ given by

$$G(x, y) = \begin{cases} \frac{u_1(x)u_2(y)}{W}, & \text{if } a \leq x < y \leq b \\ \frac{u_2(x)u_1(y)}{W}, & \text{if } a \leq y < x \leq b \end{cases}$$

is continuous. Moreover $T: f \mapsto \int_a^b G(x, y)f(y) dy$ maps $C[a, b]$ into $\mathcal{D}$

and $L[Tf] = f$ for every $f \in C[a, b]$, with $L[u] = -(pu')' + qu$.

Proof: Note that G is continuous because it is defined using continuous functions and $\lim_{\substack{x_n \rightarrow x \\ y_n \rightarrow x \\ x_n < y_n}} G(x_n, y_n) = G(x, x) = \lim_{\substack{x_n \rightarrow x \\ y_n \rightarrow x \\ y_n \geq x_n}} G(x_n, y_n)$, $\forall x \in [a, b]$.

Given $f \in C[a, b]$, let

$$u(x) := (Tf)(x) = \int_a^b G(x, y) f(y) dy = \frac{u_2(x)}{W} \int_a^x u_1(y) f(y) dy + \frac{u_1(x)}{W} \int_x^b u_2(y) f(y) dy.$$

Then, $u \in C^1[a, b]$

and $u'(x) = \frac{u_2'(x)}{W} \int_a^x u_1(y) f(y) dy + \frac{u_1'(x)}{W} \int_x^b u_2(y) f(y) dy$. So, $u' \in C^1[a, b]$,

and $u''(x) = \frac{u_2''(x)}{W} \int_a^x u_1(y) f(y) dy + \frac{u_1''(x)}{W} \int_x^b u_2(y) f(y) dy +$
 $+ \left(\frac{u_2'(x) u_1(x)}{W} f(x) - \frac{u_1'(x) u_2(x)}{W} f(x) \right) = - \frac{f(x)}{p(x)}.$

Thus $u'' \in C[a, b]$, so in all, $u \in C^2[a, b]$. Furthermore, $\rightarrow$ split up

$$-(p(x)u'(x))'(x) + q(x)u(x) = -p'(x)u'(x) - p(x)u''(x) + q(x)u(x)$$

$$= \left(-p'(x) \frac{u_2'(x)}{W} - p(x) \frac{u_2''(x)}{W} + q(x) \frac{u_2(x)}{W} \right) \int_a^x u_1(y) f(y) dy +$$

$$+ \left(-p'(x) \frac{u_1'(x)}{W} - p(x) \frac{u_1''(x)}{W} + q(x) \frac{u_1(x)}{W} \right) \int_x^b u_2(y) f(y) dy + (-p(x)) \cdot \left(- \frac{f(x)}{p(x)} \right).$$

$\rightarrow = 0$

$= f(x)$, showing that $L[u] = f$, because $u \in D$ since

$$u(a) = u_1(a) \cdot \int_a^b \frac{u_2(y) f(y)}{W} dy = 0$$

and $u(b) = u_2(b) \cdot \int_a^b \frac{u_1(y) f(y)}{W} dy = 0.$

$\square$

Remark: It is possible to find $\{u_1, u_2\}$ by solving the system

$$u' = v,$$

$$v' = -\frac{p'}{p}v + \frac{q}{p}u \quad \text{first with boundary conditions } u_1(a)=0, v_1(a)=1, \\ \text{then } u_2(b)=0, v_2(b)=1.$$

[The choice $u_1(a)=1, v_1(a)=0$ in $[R]$ seems problematic.]

Example: $L[u] = -u''$

Case $[a, b] = [0, 1]$. Need to solve $-y'' = 0$ with boundary conditions above.

(NB: Not in $\mathcal{D}$.)

General solution: $y(x) = Ax + B, A, B \in \mathbb{R}$.

For $y(0)=0, y'(0)=1$, get $B=0, A=1$, so take $u_1(x) = x$.

$y(1)=0, y'(1)=1$, get $A+B=0, A=1$ so $B=-1$ thus $u_2(x) = x-1$.

Get Wronskian $W = \underset{\substack{\uparrow \\ p(x)=1, \forall x}}{u_1'(x)u_2(x) - u_2'(x)u_1(x)} = 1 \cdot (x-1) - 1 \cdot x = -1$,

$$\text{and } G(x, y) = \begin{cases} -x(y-1) & \text{if } 0 \leq x < y \\ -(x-1)y & \text{if } y \leq x \leq 1 \end{cases} = \begin{cases} x(1-y) & \text{if } 0 \leq x < y \leq 1 \\ (1-x)y & \text{if } 0 \leq y \leq x \leq 1. \end{cases}$$

Case $[a, b] = [0, \pi]$, see [B, Example 5.4.4], but note that there is

a sign difference in the definition of the Wronskian, so that in [B], for this example we get $W = \pi$ (and not $-\pi$, as defined in [R]).

However, the Green's function in [B] has $-\frac{1}{W}$, so in all it is the same as in [R].

$$G(x, y) = \frac{1}{\pi} \begin{cases} x(\pi - y), & \text{if } 0 \leq x \leq y < \pi, \\ y(\pi - x), & \text{if } 0 \leq y \leq x \leq \pi. \end{cases}$$

Fine, so how do we solve $L[u] = \lambda u$?

We need a compact, self-adjoint operator on $L^2[a, b]$. So far, we have a linear operator $T: C[a, b] \rightarrow L^2[a, b]$.

Fact: (See, e.g., [R, Lemma 17.6]). The space $\mathcal{D}$ is dense in $L^2[a, b]$.

(The proof uses Weierstrass' theorem asserting that the ^{space of} polynomials $P[a, b]$ is dense in $C[a, b]$ in the $\|\cdot\|_\infty$ -norm. Since $[a, b]$ is a compact set, $P[a, b]$ is dense in $C[a, b]$ with respect to the 2-norm. However,

$C[a, b]$ is dense in $L^2[a, b]$ with respect to the 2-norm, so also $P[a, b]$ is dense in $L^2[a, b]$. Use elements in $P[a, b]$ to approximate any element in $\mathcal{D}$ by elements in $C[a, b]$ in $\|\cdot\|_2$).

Theorem: Assume that $L[u] = -(pu')' + qu$ is injective as a linear map $L: \mathcal{D} \rightarrow C[a, b]$. Define $T: L^2[a, b] \rightarrow L^2[a, b]$ by extending $T: C[a, b] \rightarrow \mathcal{D}$,

$$\text{so } T f(x) := \int_a^b G(x, y) f(y) dy \quad \text{for } f \in L^2[a, b].$$

We have: (a) T is compact and self-adjoint.

(b) $\ker T = \{0\}$, that is, T is injective.

(c) The eigenvalues of T form a countable sequence converging to 0 and all the eigenvectors of T are in $\mathcal{D}$. Moreover, after suitable normalisation, these eigenvectors form an o.n.b for $L^2[a, b]$.

Proof: Note that by construction, $G(x, y) = G(y, x)$, $\forall (x, y) \in [a, b]^2$.

Thus T is s.a. and it is compact (even Hilbert-Schmidt) by earlier results. By the Spectral Theorem, its eigenvalues form a countable

sequence converging to 0 (if it is infinite). Note that T cannot have finite rank: we know that $T(C[a, b]) \subseteq \mathcal{D}$ and in fact,

$\mathcal{D} \subseteq T(L^2[a, b])$. Why? Let $u \in \mathcal{D}$. Then $g := L[u] \in C[a, b]$.

By previous result, $L(Tg) = g = L[u]$. So $L(Tg - u) = 0$ which by injectivity implies $Tg = u$. Since $Tg, u \in C[a, b]$, get $u = Tg = Tg \in T(\mathcal{H})$.

Suppose $Tf = 0$ for $f \in L^2[a, b]$. Given $g \in \mathcal{D}$, write $g = Th$,

with $h \in L^2[a, b]$. Then

$$0 = \langle Tf, h \rangle = \langle f, Th \rangle = \langle f, g \rangle \text{ so } f \in \overline{\mathcal{D}}^\perp. \text{ But } \overline{\mathcal{D}} = L^2[a, b],$$

thus $f = 0$, showing $\ker T = \{0\}$. This means $0 \notin \sigma_p(T)$.

Note: $T|_{C[a,b]} \circ L = I_{\mathcal{D}}$ and $L \circ T|_{C[a,b]} = I_{C[a,b]}$.

So L and $T|_{C[a,b]}$ are inverses of each other, as linear maps.

Note: We know from Corollary 17.2 that 0 is not an eigenvalue of the Sturm-Liouville problem, so $\ker L = \{0\}$.

Claim: $Tf \in C[a, b]$ for every $f \in L^2[a, b]$.

Proof: By uniform continuity of G on $[a, b] \times [a, b]$, for given $\varepsilon > 0 \exists \delta > 0$

so that $|x - x'| < \delta \Rightarrow |G(x, y) - G(x', y)| < \varepsilon$ for all $y \in [a, b]$
 $x, x' \in [a, b]$

Then

$$|Tf(x) - Tf(x')| \leq \int_a^b |G(x, y) - G(x', y)| \cdot |f(y)| dy \leq \varepsilon \int_a^b |f(y)| dy$$

$$= \varepsilon \langle 1, f \rangle$$

The constant function 1, which is in $L^2[a, b]$ because $[a, b]$ is bounded

$$\leq \varepsilon \cdot \|1\|_2 \cdot \|f\|_2 \leq \varepsilon (b-a)^{1/2} \cdot \|f\|_2 \text{ whenever } |x - x'| < \delta.$$

Claim If $f \in L^2[a, b]$ is an eigenvector of T , corresponding to some eigenvalue μ (which we know is $\neq 0$), then $f \in \mathcal{D}$.

Proof: From $Tf = \mu f$ get $f = \frac{1}{\mu} Tf \in C[a, b]$ by previous claim.

However, $T|_{C[a,b]} = T$, so $f = \frac{1}{\mu} Tf \in \mathcal{D}$ by Thm 17.5, as claimed.

Since $\ker T = \{0\}$, its eigenvectors form an o.n.b for $L^2[a, b]$. $\square$.