

Main result of Sturm-Liouville problem: see [R, Thm 17.8]. ①

Let $L[u] = -(p(x)u')' + q(x)u$, for $p \in C^1[a, b]$, $p(x) > 0$ for all $x \in [a, b]$ and $q \in C[a, b]$ with $q(x) \geq 0$ for all $x \in [a, b]$. Then the eigenfunctions of $L[u] = \lambda u$ in $\mathcal{D} = \{u \in C^2[a, b] \mid u(a) = u(b) = 0\}$ form an o.n.b. basis for $L^2[a, b]$.

- Moreover
- (i) every eigenvalue is strictly positive;
 - (ii) each eigenvalue is simple; and
 - (iii) the eigenvalues can be ordered to form a sequence that tends to ∞ .

Proof: Part (i) was proved in Cor. 17.2.

To (ii) Part (ii) $\rightarrow$ Cor 17.4.

For (iii), let $\{\mu_j\}_j$ be the real, non-zero eigenvalues of T .

Suppose λ is an eigenvalue of the (S-L) problem, thus $L[u] = \lambda u$ for some $u \in \mathcal{D}$, $u \neq 0$.

Then $u = T(L[u]) = \lambda T u$, so $T u = \frac{1}{\lambda} u$ (since $\ker L = \ker T = \{0\}$).

This means that the eigenvalues of (S-L) problem are $\{\frac{1}{\mu_j}\}_j$, which we know is a sequence. Since $\lim_{j \rightarrow \infty} \mu_j = 0$, we obtain claim (iii). $\square$

Observation: The Sturm-Liouville problem can be formulated with other boundary conditions, for example $u(a) = 0$, $u'(b) = 0$.

Observation To show that $\mathcal{D}$, defined in the theorem, is dense in $L^2[a, b]$, use a collection $\{\varphi_\delta\} \in C^2[a, b]$ defined as, see [R, Lemma 17.6].

$$\varphi_\delta(x) = \begin{cases} \phi((x-a)/\delta) & \text{if } a \leq x < a+\delta \\ 1 & \text{if } a+\delta \leq x < b-\delta \\ \phi((b-x)/\delta) & \text{if } b-\delta < x \leq b \end{cases}, \text{ with } \phi(x) = 3x - 3x^2 + x^3.$$

One shows: given $f \in L^2[a, b]$, find $g \in C^2[a, b]$ with $\|f - g\|_2 < \frac{\varepsilon}{2}$, then find δ small enough so that $\|\varphi_\delta g - g\|_2 < \frac{\varepsilon}{2}$. Note that $\varphi_\delta g \in \mathcal{D}$, as needed.

Example: $L[u] = -u''$, $u \in L^2(0,1)$, $u(0)=0$, $u(1)=0$.

By the Main Theorem, we know that the eigenvectors of this L form an o.n.b for $L^2(0,1)$. So, what is it?

Solve $-u'' = \lambda u$, or $u'' + \lambda u = 0$. Characteristic eq: $r^2 + \lambda = 0$.

Case: $\lambda > 0$. Then $u(x) = A e^{\sqrt{\lambda}x} + B e^{-\sqrt{\lambda}x}$. From $u(0)=0=u(1)$ get $A+B=0 = A e^{\sqrt{\lambda}} + B e^{-\sqrt{\lambda}}$ so $B=-A$ and $A e^{\sqrt{\lambda}} = A e^{-\sqrt{\lambda}}$, so $A=0$. This gives $u(x) \equiv 0, \forall x$, not possible for an eigenfunction. Thus $L[u] = \lambda u$ has no solutions when $\lambda > 0$.

Case $\lambda = 0$. Then u is a solution of $-u'' = 0$ with $u(0)=0=u(1)$. From $u(x) = Ax + B$ get $0 = B = A+B$ so $u(x) = 0, \forall x$, not possible.

Case $\lambda < 0$: Need to solve $-u'' + \lambda u = 0$ with $u(0)=0=u(1)$.

get $u(x) = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x$, thus $A=0$ and $B \sin \sqrt{\lambda} = 0$.

To have $u(x) \not\equiv 0$ need $B \neq 0$, so need $\sin \sqrt{\lambda} = 0$. Thus,

$$\sqrt{\lambda} \in \pi \mathbb{Z}_+ \text{ i.e., } \lambda \in \{(\pi k)^2 \mid k \in \mathbb{Z}_+\}.$$

Want $u(x) = \sin(\pi k x)$ to have $\|u(x)\|_2 = 1$.

$$\int_0^1 \sin(\pi k x) \sin(\pi k x) dx = -\frac{1}{\pi k} \underbrace{\sin(\pi k x) \cdot \cos(\pi k x)}_{\equiv 0} \Big|_0^1 + \frac{1}{\pi k} (\cos(\pi k x)) \Big|_0^1$$

$$+ \int_0^1 \pi k \cdot \cos(\pi k x) \cdot \frac{1}{\pi k} (\cos(\pi k x)) dx = \int_0^1 \cos^2(\pi k x) dx$$

$$= 1 - \int_0^1 \sin^2(\pi k x) dx, \text{ thus } \int_0^1 \sin^2(\pi k x) dx = \frac{1}{2}.$$

$$\Rightarrow \text{o.n.b. } \left\{ \frac{1}{\sqrt{2}} \sin(k\pi x) \right\}_{k=1}^{\infty}$$

[B, Oppgave 4.17] La $\mathcal{H} = \ell^2(\mathbb{N})$ og $\lambda \in \ell^2(\mathbb{N})$ være gitt ved

$\lambda(n) = \frac{1}{n}$ for alle $n \in \mathbb{N}$. Set $M_\lambda \in \mathcal{B}(\mathcal{H})$ være multiplikasjonsoperatoren, $M_\lambda: (x_1, x_2, x_3, \dots) \mapsto (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots)$.

Vel, at $M_\lambda \in \mathcal{K}(\mathcal{H})$.

Vis, at $\overline{M_\lambda(\mathcal{H})} = \mathcal{H}$ men at $M_\lambda(\mathcal{H}) \neq \mathcal{H}$.

Løsning: La $\{y_n\}_n \in \mathcal{H}$. For hvert $N \geq 1$, sett

$$y^N = (y_1, 2y_2, \dots, Ny_N, 0, 0, \dots). \text{ Da er } y^N \in \ell^2, \text{ og}$$

$$M_\lambda y^N = (y_1, y_2, \dots, y_N, 0, 0, \dots).$$

$$\text{Dermed er } \lim_{N \rightarrow \infty} \|M_\lambda y^N - y\|_2^2 = \lim_{N \rightarrow \infty} \left| \sum_{k=N+1}^{\infty} |y_k|^2 \right|$$

$$\leq \lim_{N \rightarrow \infty} \sum_{k=N+1}^{\infty} |y_k|^2 = 0.$$

At $M_\lambda(\mathcal{H}) \neq \mathcal{H}$ ses for eksempel ved at

$\lambda(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{N}, \dots) \in \ell^2$ men $\lambda \notin M_\lambda(\mathcal{H})$ siden eneste valget er $\lambda = M_\lambda(1, 1, 1, \dots)$ og $\{x_n\}_n = \{1\} \notin \ell^2$.

[B, Oppgave 4.18].

La $\mathcal{H}$ et Hilbertrom. Hvis $T \in \mathcal{B}(\mathcal{H})$ oppfyller at det finnes $\alpha > 0$ s.a. $\|Tx\| \geq \alpha\|x\|$ for alle $x \in \mathcal{H}$ (kalles T nedad begrenset) da er $T(\mathcal{H})$ lukket.

Løsning: Anta at slik α finnes. La $\{y_n\} \in T(\mathcal{H})$ konvergere mot $y \in \mathcal{H}$. Vi skal vise, at $y \in T(\mathcal{H})$. Vi har $y_n = Tx_n$ for hvert $n \in \mathbb{N}$. For $n \geq m$ i $\mathbb{N}$ er

$\|Tx_n - Tx_m\| \geq \alpha\|x_n - x_m\|$. Siden $\{Tx_n\}_n$ er konvergent, er den Cauchy, og folgelig er $\{x_n\}_n$ en Cauchy folge, dermed konvergent. La $x = \lim_n x_n$. Da T er begrenset vil $Tx = \lim_n Tx_n$. Dermed er $Tx = y \in T(\mathcal{H})$. $\square$

Oppgaver for 8. oktober:

8/10/2025 (4)

[R, 15.3, 15.9 (see example 11.7), 16.6, 16.7(i)]

[B, 4.17, 4.18].

[R, 15.3] $T: \ell^2 \rightarrow \ell^2, T: (x_1, x_2, x_3, \dots) \mapsto (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots)$ er kompakt.

Merk: La $f = (\frac{1}{j})_{j \geq 1} \in \ell^\infty, \|f\|_\infty = 1$. La $e^j = \{e_k^j\}_{k \geq 1}$ være den kanoniske basisen i ℓ^2 . Da er $T = M_f$. Siden $\lim_{j \rightarrow \infty} \frac{1}{j} = 0$,
er $T \in K(\ell^2)$ av Eksempel 4.1.8 i [B].

[R, 15.9] La $T': \ell^2 \rightarrow \ell^2$ være gitt som $T': (x_1, x_2, x_3, \dots) \mapsto (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots)$.
Vis, at T' er kompakt og at den ikke har noen egenverdier.

Løsning: La $\Delta_r: \ell^2 \rightarrow \ell^2$ være skift-til-høyre operatoren

$$\Delta_r: (x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, \dots).$$

Det gjelder, at $T'(\{x_n\}) = (\Delta_r \circ T)(\{x_n\})$, for alle $\{x_n\}_n \in \ell^2$,

så $T' = \Delta_r \circ T$ i $B(\ell^2)$. Siden T er kompakt, det følger at
 $\Delta_r \circ T \in K(\ell^2)$, med andre ord er T' kompakt.

Antar at $\lambda \in \mathbb{C} \setminus \{0\}$ er en egenverdi for T' . Det eksisterer så
 $\{x_n\} \neq 0$ i ℓ^2 slik at $T'(\{x_n\}) = \lambda \{x_n\}$, altså

$$(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots)$$

Da er $\lambda x_1 = 0$, så $x_1 = 0$. Videre er $x_1 = \lambda x_2, \dots, \frac{x_k}{k} = \lambda \frac{x_{k+1}}{k+1}, \dots$

Ved induksjonsprinsippet er $x_k = 0$, for alle $k \geq 1$. Dette er et
motstrid, så $\sigma_p(T) \cap (\mathbb{C} \setminus \{0\}) = \emptyset$.

Merk at $0 \notin \sigma_p(T)$ siden T er injektiv. Dette ses lett da $\ker T = \{0\}$.

∴ alt er $\sigma_p(T) = \emptyset$.

8/10/2025 (5)

Oppgave [R, 16.6].

La $\{e_j(x)\}$ være en o.n.mengde i $H = L^2([a, b], \mathbb{R})$, la $\{\lambda_j\}_{j \in \mathbb{N}}$ være en følge i $\mathbb{R}$, og sett

$$K(x, y) = \sum_{j=1}^{\infty} \lambda_j e_j(x) e_j(y), \quad (x, y) \in [a, b] \times [a, b].$$

Sett $(Tu)(x) = \int_a^b K(x, y) u(y) dy$. Vis at e_k er en egenvektor

til T hørende til egenverdien λ_k , for hvert $k \in \mathbb{N}$.

Løsning Først, vi har $K(\cdot, \cdot) \in L^2([a, b] \times [a, b])$, siden ved Tonelli's,

$$\begin{aligned} \int_a^b \int_a^b |K(x, y)|^2 dx dy &\leq \int_a^b \int_a^b \sum_{j=1}^{\infty} |\lambda_j e_j(x) e_j(y)|^2 dx dy \leq \\ &\leq \sum_{j=1}^{\infty} |\lambda_j|^2 \left(\int_a^b |e_j(x)|^2 dx \right) \left(\int_a^b |e_j(y)|^2 dy \right) = \sum_{j=1}^{\infty} |\lambda_j|^2 < \infty \end{aligned}$$

Dermed er funksjonen $K_x : [a, b] \rightarrow \mathbb{R}$, $K_x(y) = K(x, y) \in L^2([a, b])$,

siden $\int_a^b |K(x, y)|^2 dy < \infty$ m.o. Spesielt er $K_x \in L^1[a, b]$, siden

$$\int_a^b |K_x(y)| dy \leq \|K_x\|_2 \cdot \|1_{[a, b]}\|_2 < \infty. \text{ Ved dominert konvergens, ser}$$

$$\begin{aligned} \text{vi at } (Tu)(x) &= \int_a^b \sum_{j=1}^{\infty} \lambda_j e_j(x) e_j(y) u(y) dy = \sum_{j=1}^{\infty} \lambda_j e_j(x) \int_a^b e_j(y) u(y) dy \\ &= \sum_{j=1}^{\infty} \lambda_j e_j(x) \langle u, e_j \rangle \\ &= \sum_{j=1}^{\infty} \lambda_j \langle u, e_j \rangle e_j(x), \text{ så } Tu = \sum_{j=1}^{\infty} \lambda_j \langle u, e_j \rangle e_j, \text{ for} \end{aligned}$$

hvert $u \in H = L^2[a, b]$.

Herav følger det, at $Te_k = \lambda_k e_k$, $\forall k \in \mathbb{N}$. Hvis $u \in H$ slik at $Tu = \lambda u$ og $\lambda \neq \lambda_j, \forall j$, da er $\sum (\lambda_j - \lambda) \langle u, e_j \rangle e_j = 0$, så $\langle u, e_j \rangle = 0, \forall j \Rightarrow u = 0$.