

Duals of Banach spaces

Recall def: If X is a normed space over $\mathbb{K}$ (equal to $\mathbb{R}$ or $\mathbb{C}$), then its dual space, denoted X^* , is $\mathcal{B}(X, \mathbb{K})$. In other words,

$$X^* = \{ \phi: X \rightarrow \mathbb{K} \mid \phi \text{ linear and bounded} \}$$

By definition of the norm in $\mathcal{B}(X, \mathbb{K})$, for $\phi \in X^*$ we have.

$$\|\phi\| = \sup \{ |\phi(x)| \mid x \in X, \|x\| \leq 1 \} \stackrel{\text{short notation}}{=} \sup_{\|x\| \leq 1} |\phi(x)|$$

Fact/Theorem If X is any normed space, its dual X^* is always a Banach space.

Proof: See MAT3400/4400. The key property is: the target space $\mathbb{K}$ is a complete normed space (a Banach space).

Question: Given a normed space X , can we describe its dual space X^* ? Do we even know X^* is nonempty?

① Case: $X = \mathcal{H}$ a Hilbert space. Suppose $\mathbb{K} = \mathbb{C}$.

By the Riesz lemma, the map $\bar{\Phi}: \mathcal{H} \rightarrow \mathcal{H}^*$, $y \mapsto \bar{\Phi}_y$ where

$\bar{\Phi}_y(x) = \langle x, y \rangle$ for all $x \in \mathcal{H}$ is well-defined ($\bar{\Phi}_y \in \mathcal{H}^*$) and determines an isometric ($\|\bar{\Phi}_y\| = \|y\|$, $\forall y \in \mathcal{H}$) conjugate linear bijection (the non-trivial part is surjectivity of $\bar{\Phi}$). This means that every ϕ in $\mathcal{H}^*$ is of the form $\phi = \bar{\Phi}_y$ for some $y \in \mathcal{H}$, which is necessarily unique by injectivity of $\bar{\Phi}$. So $\mathcal{H}^*$ is always $\neq \emptyset$.

[Note: ϕ is $\backslash \phi$, $\bar{\Phi}$ is $\backslash \Phi$].

② Case $X = \ell^p(\mathbb{N}) = \{ (x_n)_{n \in \mathbb{N}} \mid x_n \in \mathbb{K}, \forall n \in \mathbb{N}, \sum_n |x_n|^p < \infty \}$

with $\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$, where $1 < p < \infty$. Let q so $\frac{1}{p} + \frac{1}{q} = 1$.

Then $(\ell^p(\mathbb{N}))^*$ is identified, up to isometric isomorphism, with $\ell^q(\mathbb{N})$.

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Some terminology:

Def: Let X, Y be normed spaces, and $L: X \rightarrow Y$ a mapping. We say

- (i) L is isometric if $\|L(x)\|_Y = \|x\|_X, \forall x \in X$;
- (ii) L is an isomorphism if L is linear, bijective (injective & surjective), and both L and L^{-1} are continuous ($\Leftrightarrow$ bounded). Write $X \cong Y$ if such L exists.

Theorem For $1 < p < \infty$ real number, let q be the unique positive real number s.t. $\frac{1}{p} + \frac{1}{q} = 1$ (call such p, q for dual exponents). Then map

(a) The map $L: \ell^2(\mathbb{N}) \rightarrow (\ell^p(\mathbb{N}))^*$, $L((y_n)_n)((x_n)_n) = \sum_{n \in \mathbb{N}} x_n y_n$ is an isometric isomorphism.

(b) If (Ω, Σ, μ) is a measure space, then the map

$L: L^2(\mu) \rightarrow (L^p(\mu))^*$, $L(g)(f) = \int_X fg d\mu, g \in L^2(\mu), f \in L^p(\mu)$, is an isometric isomorphism.

(c) The map $L: \ell^\infty(\mathbb{N}) \rightarrow (\ell^1(\mathbb{N}))^*$, $L((y_n)_n)((x_n)_n) = \sum_n x_n y_n$, is an isometric isomorphism.

(d) If (Ω, Σ, μ) is a σ -finite measure space, the map $L: L^\infty(\mu) \rightarrow (L^1(\mu))^*$, $L(g)(f) = \int_\Omega fg d\mu, g \in L^\infty(\mu), f \in L^1(\mu)$, is an isometric isomorphism.

Proof: See MAT 3400/4400.

However, $(\ell^\infty(\mathbb{N}))^* \neq \ell^1(\mathbb{N}), (L^\infty(\mu))^* \neq L^1(\mu)$.

We need the Hahn-Banach theorem and some consequences of it.

Recall from MAT 3400-4400:

- (A) Theorem (The Hahn-Banach theorem for real vector spaces).
 Suppose that V is a real vector space and $p: V \rightarrow \mathbb{R}$ is a map that is subadditive, meaning

$$p(u+v) \leq p(u) + p(v), \quad \forall u, v \in V,$$

and positive homogeneous, meaning

$$p(\alpha v) = \alpha p(v) \text{ for all } v \in V \text{ and } \alpha \in \mathbb{R} \geq 0.$$

If V_0 is a subspace of V and $l_0: V_0 \rightarrow \mathbb{R}$ is a linear functional s.t.

$$l_0(v) \leq p(v) \text{ for all } v \in V_0,$$

- (Then there exists a linear functional $l: V \rightarrow \mathbb{R}$ such that $l(v) = l_0(v)$ for all $v \in V_0$ and $l(v) \leq p(v)$ for all $v \in V$.)

Proof: See MAT 3400/4400. We write $l|_{V_0} = l_0$ and $l \leq p$ in this case.

- (B) Theorem (The Hahn-Banach theorem for complex vector spaces).

Suppose that V is a complex vector space and $p: V \rightarrow \mathbb{R}$ is a map that is:

- (p is a seminorm) $\left\{ \begin{array}{l} \text{(i) non-negative, meaning } p(v) \geq 0, \forall v \in V; \\ \text{(ii) subadditive;} \\ \text{(iii) homogeneous, meaning } p(\alpha v) = |\alpha| p(v), \forall \alpha \in \mathbb{C}, \forall v \in V. \end{array} \right.$

Let $V_0 \subseteq V$ be a subspace and $l_0: V_0 \rightarrow \mathbb{C}$ be a linear functional s.t.

$$|l_0(w)| \leq p(w) \text{ for all } w \in V_0.$$

Then there exists a linear functional $l: V \rightarrow \mathbb{C}$ s.t. $l|_{V_0} = l_0$ and $|l| \leq p$.

If V is a normed space, then any $\phi \in V_0^*$ can be extended to some $l \in V^*$ with the same norm.

Note: A map p satisfying (i)-(iii) is called a seminorm on V .

Thm. (B) will follow from thm (A) using "complexification" of real functionals.

Notation: for a vector space V over $\mathbb{C}$, we let $V_{\mathbb{R}}$ denote the same space, seen as a vector space over $\mathbb{R}$.

Lemma Let V be a complex vector space.

(a) Given a linear functional $l: V \rightarrow \mathbb{C}$, define $l_{\mathbb{R}}: V_{\mathbb{R}} \rightarrow \mathbb{R}$ by $l_{\mathbb{R}}(v) = \operatorname{Re}(l(v))$ for every $v \in V$.

Then $l_{\mathbb{R}}$ is linear (on the real vector space $V_{\mathbb{R}}$) and

$$(*) \quad l(v) = l_{\mathbb{R}}(v) - i l_{\mathbb{R}}(iv), \text{ for all } v \in V.$$

Further, $l_{\mathbb{R}}$ is the unique $\mathbb{R}$ -linear functional on $V_{\mathbb{R}}$ satisfying $(*)$.

If $p: V \rightarrow \mathbb{R}$ is a seminorm and $|l(v)| \leq p(v)$ for all $v \in V$, then also $|l_{\mathbb{R}}(v)| \leq p(v)$, $\forall v \in V_{\mathbb{R}}$. In case V is a normed space and

l is bounded (i.e., $l \in V^*$), then $l_{\mathbb{R}}$ is bounded (i.e., $l_{\mathbb{R}} \in V_{\mathbb{R}}^*$) and $\|l\| = \|l_{\mathbb{R}}\|$.

(b) Conversely, if $\phi: V_{\mathbb{R}} \rightarrow \mathbb{R}$ is linear, then $l: V \rightarrow \mathbb{C}$ given by

$$(**) \quad l(v) = \phi(v) - i \phi(iv) \text{ for } v \in V$$

is linear. In case $|\phi(v)| \leq p(v)$ for all $v \in V$, also $|l(v)| \leq p(v)$ for all $v \in V$, where p is some seminorm. Finally, if V is normed space and $\phi \in V_{\mathbb{R}}^*$, then $l \in V^*$ and $\|l\| = \|\phi\|$.

Proof: (a) Let $l: V \rightarrow \mathbb{C}$ be linear, define $l_{\mathbb{R}}$ as above. Set also

$l_{\mathbb{I}}: V_{\mathbb{R}} \rightarrow \mathbb{R}$ be $l_{\mathbb{I}}(v) = \operatorname{Im}(l(v))$ for $v \in V$. Given $v \in V$, we

have $l(v) = l_{\mathbb{R}}(v) + i l_{\mathbb{I}}(v)$. Since v is $\mathbb{C}$ -linear, $l(iv) = i l(v)$, so

$$l_{\mathbb{R}}(iv) + i l_{\mathbb{I}}(iv) = -l_{\mathbb{I}}(v) + i l_{\mathbb{R}}(v). \text{ Thus } l_{\mathbb{R}}(iv) = -l_{\mathbb{I}}(v),$$

which implies $(*)$. Clearly $l_{\mathbb{R}}$ is $\mathbb{R}$ -linear. If $\tilde{l}: V_{\mathbb{R}} \rightarrow \mathbb{R}$ satisfies

$l(v) = \tilde{l}(v) - i \tilde{l}(iv)$ for all $v \in V$, then $l(v) = l_{\mathbb{R}}(v)$ for all $v \in V$ (hence

$\tilde{l}(v), \tilde{l}(iv), l_{\mathbb{R}}(v), l_{\mathbb{R}}(iv) \in \mathbb{R}$. Since $|l_{\mathbb{R}}(v)| \leq |l(v)|$ for all $v \in V$,

we get $|l_{\mathbb{R}}(v)| \leq p(v)$ if $|l| \leq p$. (Note: $|\operatorname{Re}(z)| \leq |z|$ for all $z \in \mathbb{C}$.)

Assume V is a normed space and $l \in V^*$.

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Since $|l_{\mathbb{R}}(x)| \leq |l(x)|$ for all $x \in V$, we get $\|l_{\mathbb{R}}\| \leq \|l\|$. To show $=$, let $x \in V$ and take $\alpha \in \mathbb{C}$ so that $|l(x)| = \alpha l(x)$. [$\alpha = e^{i\theta}$, some $\theta \in \mathbb{R}$]
Then $|\alpha| \neq 0$ (assuming $l(x) \neq 0$). By $\mathbb{C}$ -linearity of l , we get

$$\underbrace{|l(x)|}_{\substack{\uparrow \\ \mathbb{R}}} = \alpha l(x) = l(\alpha x) \underset{\substack{\uparrow \\ \text{since } l_{\mathbb{R}}(\alpha x) = 0}}{=} l_{\mathbb{R}}(\alpha x) \leq \|l_{\mathbb{R}}\| \cdot \|\alpha x\| = \|l_{\mathbb{R}}\| \cdot \underbrace{|\alpha|}_{=1} \cdot \|x\|$$

Taking supremum when $\|x\| \leq 1$, we get $\|l\| \leq \|l_{\mathbb{R}}\|$ so equality.

(c). Let ϕ be $\mathbb{R}$ linear, define l by $(*)$. Need $l(\alpha v + \beta w) = \alpha l(v) + \beta l(w)$ for all $\alpha, \beta \in \mathbb{C}$. First, we show

$$l(iw) = i l(w), \text{ for all } w \in V.$$

$$\begin{aligned} \text{But } l(iw) &\stackrel{(*)}{=} \phi(iw) - i \phi(i^2 w) = \phi(iw) - i \phi(-w) \\ &= \phi(iw) + i \phi(w) \text{ by } \mathbb{R}\text{-linearity of } \phi \\ &= i(\phi(w) - i \phi(iw)) = i l(w) \end{aligned}$$

Then $l(a+ib)w = (a+ib)l(w)$ for all $a, b \in \mathbb{R}$ and $\mathbb{C}$ -linearity follows

Assume $|\phi(v)| \leq p(v)$, $\forall v \in V$.

Let $v \in V$. If $l(v) = 0$, then $|l(v)| \leq p(v) = 0$. For $l(v) \neq 0$, set

$$\alpha = \frac{\overline{l(v)}}{|l(v)|}. \text{ Then } |\alpha| = 1 \text{ and } |l(v)| = \alpha l(v) \underset{\substack{\uparrow \\ l \text{ is linear}}}{=} l(\alpha v) = \phi(\alpha v), \text{ } \underset{\substack{\uparrow \\ l(\alpha v) \in \mathbb{R}}}{}$$

so $|l(v)| = |\phi(\alpha v)| \leq p(\alpha v) = |\alpha| p(v) = p(v)$, as needed.

That $\|l\| = \|\phi\|$ follows as in (a).

Proof of Thm B (Hahn-Banach theorem, complex case). Let p, V_0, l_0 be given as in the hypotheses of the theorem. Let $l_{0, \mathbb{R}}: (V_0)_{\mathbb{R}} \rightarrow \mathbb{R}$ be as in Lemma(a), so that $\|l_{0, \mathbb{R}}\| \leq p$. By the Hahn-Banach theorem for real spaces, there exists an extension $\phi: V_{\mathbb{R}} \rightarrow \mathbb{R}$ of $l_{0, \mathbb{R}}$ with $|\phi| \leq p$. By Lemma(b), we get $l: V \rightarrow \mathbb{C}$ complex linear with $\|l\| \leq p$. Check that $l|_{V_0} = l_0$ (use $(*)$). If V is a normed space, get $\|l\| = \|l_0\|$ by Lemma. □