

Some applications of the Hahn-Banach theorem

Lemma (Existence of support functional)

If X is a normed space, then for every $x \in X \setminus \{0\}$ there exists $\phi \in X^*$ with $\|\phi\|=1$ and $\phi(x)=\|x\|$.

Proof: Let $X_0 = \mathbb{C}x = \{\alpha x \mid \alpha \in \mathbb{C}\}$ and $\phi_0: X_0 \rightarrow \mathbb{C}$ be given by

$$\phi_0(\alpha x) = \alpha \|x\|. \text{ With } p: X \rightarrow \mathbb{C}, p(x) = \|x\|, \text{ we have}$$

$$|\phi_0(\alpha x)| = |\alpha| \|x\| = \|\alpha x\| = p(\alpha x), \forall \alpha x \in X_0. \text{ Clearly, } \|\phi_0\| \leq 1.$$

The Hahn-Banach theorem provides an extension ϕ of ϕ_0 with $\|\phi\| = \|\phi_0\|$.

Since $x \neq 0$, with $\alpha = \frac{1}{\|x\|}$ we have $|\phi_0(\alpha x)| = |\alpha| \cdot \|x\| = 1 \geq \|\phi_0\|$, so $\|\phi_0\|=1$, hence $\|\phi\|=1$.

Corollary (Separation of points)

If $x \neq y$ in a normed space, there exists $\phi \in X^*$ with $\phi(x) \neq \phi(y)$ and $\|\phi\|=1$.

Proof: Apply previous lemma with $x = x_1 - x_2$.

Theorem (Existence of separating functional).

Let X be a normed space and M a closed subspace. For every $x \in X \setminus M$ there exists $\phi \in X^*$ with $\|\phi\|=1$, $\phi(x) = \text{dist}(x, M) = \inf\{\|x - y\| \mid y \in M\}$ and $\phi(y) = 0, \forall y \in M$.

Proof: Let $x \in X \setminus M$ and set $X_0 = \text{span}\{M, x\} = \{m + \alpha x \mid m \in M, \alpha \in \mathbb{C}\}$.

Since M is closed, $\delta := \text{dist}(x, M) > 0$. Define $l_0: X_0 \rightarrow \mathbb{C}$ by $l_0(m + \alpha x) = \alpha \delta$. Then l_0 is linear. For $m \in M$ and $\alpha \neq 0$, we have

$$|l_0(m + \alpha x)| = |\alpha| \delta \leq |\alpha| \cdot \|x + \frac{m}{\alpha}\| = \|\alpha x + m\| = p(\alpha x + m), \text{ so } |l_0| \leq p.$$

and $\|l_0\| \leq 1$. We claim that $\|l_0\| \geq 1$. For each $n \geq 1$, find $y_n \in M$ with

$$\|x - y_n\| < \frac{\delta}{1 - \frac{1}{n}} \text{ [since } \delta(1 - \frac{1}{n})^{-1} > \delta], \text{ so } \delta = l_0(x) > (1 - \frac{1}{n})\|x - y_n\|.$$

As $l_0(y_n) = 0$, we get $(1 - \frac{1}{n})\|x - y_n\| < l_0(x - y_n) \leq \|l_0\| \cdot \|x - y_n\|$. Since $x \notin M$, we can divide by $\|x - y_n\| \neq 0$. Thus, we get $1 - \frac{1}{n} \leq \|l_0\|$, so $\|l_0\| = 1$. The Hahn-Banach theorem provides an extension $\phi \in X^*$ with $\|\phi\| = \|l_0\| = 1$ and $\phi(x) = l_0(x) = \delta = \text{dist}(x, M)$. Also, $\phi|_M = l_0|_M = 0$. $\square$

Theorem If X is a normed space such that X^* is separable, then X is separable.

Proof: Assume X^* is separable, thus there exists a dense sequence $\{\phi_n\}_n$ in X^* .

For each $n \geq 1$ choose $x_n \in X$ with $\|x_n\|=1$ and $|\phi_n(x_n)| \geq \frac{\|\phi_n\|}{2}$.

Claim $\overline{\text{span}\{x_n\}} = X$ (or $\text{clim}\{x_n\} = X$, in notation from [R]).

Suppose not, then there exists $x_0 \in X \setminus \overline{\text{span}\{x_n\}}$. By previous result, there exists $\phi \in X^*$ with $\|\phi\|=1$, $\phi(x_0) = d(x_0, \overline{\text{span}\{x_n\}})$ and

$$\phi|_{\overline{\text{span}\{x_n\}}} = 0.$$

For each $n \geq 1$, $|\phi_n(x_n)| = |\phi_n(x_n) - \phi(x_n)| = |(\phi_n - \phi)(x_n)| \leq \|\phi_n - \phi\|$

Thus $\frac{1}{2}\|\phi_n\| \leq \|\phi_n - \phi\|$ for all $n \geq 1$.

Since $\{\phi_n\}_n$ is dense in X^* , there is for each $k \geq 1$

an $\phi_{n_k} \in X^*$ s.t. $\|\phi_{n_k} - \phi\| < \frac{1}{2^{k+1}}$. Therefore

$\frac{1}{2}\|\phi_{n_k}\| < \frac{1}{2^{k+1}}$ so $\lim \|\phi_{n_k}\| = 0$. But $\lim \|\phi_{n_k}\| = \|\phi\| = 1$, a contradiction.

From the claim we get a countable dense set (of finite linear combinations with rational coefficient of elements in $\{x_n\}_{n \geq 1}$).

Corollary: $(\ell^\infty(\mathbb{N}))^* \not\cong \ell^1(\mathbb{N})$.

Indeed, $\ell^1(\mathbb{N})$ is separable ($\{\sum_{j \in F} r_j e^j, r_j \in \mathbb{Q}, F \text{ finite}\}$ is dense),

but $\ell^\infty(\mathbb{N})$ is not dense. (Exercise. Hint: find an uncountable family of disjoint open sets in $\ell^\infty(\mathbb{N})$. How? Consider points

$x = \{x_n\}_n \in \ell^\infty(\mathbb{N})$ where $x_n \in \{0, 1\}$, only finitely many 1's).

Quotients of Banach spaces

Given X normed space and $M \subseteq X$ a closed subspace, we form the quotient space

$X/M = \{x+M \mid x \in X\}$. The elements are consisting of equivalence classes (or cosets) for the relation $x \sim y$ in $X \iff x-y \in M$.

Then X/M is a linear space with operations:

$$(x+M) + (y+M) = (x+y)+M,$$

$$\alpha(x+M) = \alpha x + M, \quad \text{with } \alpha \in \mathbb{C}, x, y \in X.$$

Further, X/M is a normed space with norm

$$\|x+M\| = \inf \{\|x+w\| \mid w \in M\} = \inf \{\|x-w\| \mid w \in M\}.$$

Proposition: If X is a Banach space and $M \subseteq X$ is a closed subspace, then X/M is a Banach space.

Proof: Suppose $\{x_n + M\}_{n \geq 1}$ is a Cauchy sequence in X/M .

Write $z_n = x_n + M$. Extract a subsequence $\{z_{n_k}\}$ such that

$$\forall k \geq 1, \|z_{n_{k+1}} - z_{n_k}\| < \frac{1}{2^k}. \quad (\text{Induction \& the Cauchy property})$$

Next, use induction to choose a subsequence $\{x'_k\}$ in X s.t.

$$z_{n_k} = x'_k + M \text{ and}$$

$$\|x'_{k+1} - x'_k\| < \frac{1}{2^{k-1}}.$$

Deduce that $(x'_k)_k$ is a Cauchy sequence in X , hence

convergent to some $x \in X$. Verify that $\{z_{n_k}\}$ converges

to $z := x + M$. Now, if a subsequence of a Cauchy sequence converges, then the sequence itself converges, as wanted. $\square$

QUOTIENT BANACH SPACES

Suppose that X is a normed space and Y is a non-zero subspace. Let X/Y define the set of cosets $x + Y$ with $x \in X$. Define addition of cosets by $(x+Y) + (z+Y) = (x+z) + Y$ for $x, z \in X$, and multiplication by scalars as follows $a(x+Y) = ax + Y$. On the vector space X/Y define a function by $\|x+Y\| = \inf\{\|x-y\| \mid y \in Y\}$. Show that if Y is a closed subspace of X , then this function is a norm. Moreover, show that if X is a Banach space then so is X/Y with this norm.

We only prove the last statement. Write $z_n = [x_n]$ for each n . Suppose that $\{z_n\}_{n \geq 1}$ is a Cauchy sequence in X/Y . For each $n \geq 1$ write z_n in the form $z_n = x_n + Y$ for some representative x_n in X .

Choose a subsequence z_{n_k} of z_n such that for all $k \geq 1$:

$$(1) \quad \|z_{n_{k+1}} - z_{n_k}\| < \frac{1}{2^k};$$

(use induction on k and the Cauchy property for $\varepsilon = 1/2^k$).

Next we will use induction on k to choose a sequence $\{x'_k\}$ in X such that $z_{n_k} = x'_k + Y$ for all k and

$$(2) \quad \|x'_{k+1} - x'_k\| < \frac{1}{2^{k-1}}.$$

At $k = 1$ we use (1) and the definition of the norm of the element $z_{n_{k+1}} - z_{n_k}$ in X/Y as the infimum over $y \in Y$ of the norms $\|x_{n_{k+1}} - x_{n_k} - y\|$ in X : namely we find an element $y_1 \in Y$ such that

$$\|x_{n_2} - x_{n_1} - y_1\| < \|z_{n_2} - z_{n_1}\| + \frac{1}{2} < 1;$$

we let $x'_1 = x_{n_1}$ and $x'_2 = x_{n_2} - y_1$. Note that since $y_1 \in Y$ we have $x'_2 + Y = x_{n_2} + Y = z_{n_2}$.

Assume we have $x'_1, \dots, x'_k$ with the properties that $z_{n_j} = x'_j + Y$ and $\|x'_j - x'_{j-1}\| < \frac{1}{2^{j-2}}$ for $1 \leq j \leq k$. Since

$$\|z_{n_{k+1}} - z_{n_k}\| = \inf\{\|x_{n_{k+1}} - x'_k - y\| \mid y \in Y\}$$

we can find an element y_{k+1} in Y such that

$$\|x_{n_{k+1}} - x'_k - y_{k+1}\| \leq \|z_{n_{k+1}} - z_{n_k}\| + \frac{1}{2^k} \leq \frac{1}{2^k} + \frac{1}{2^k} = \frac{1}{2^{k-1}}.$$

We let $x'_{k+1} = x_{n_{k+1}} - y_{k+1}$.

Thus we have a sequence $\{x'_k\}$ in X which satisfies (2), so it is a Cauchy sequence. Since X is a Banach space, $\{x'_k\}$ is convergent to an element x in X .

We claim that $x + Y$ is a limit for the subsequence $\{z_{n_k}\}$; indeed, we use that

$$\|z_{n_k} - (x + Y)\| \leq \|x'_k - x\|,$$

so convergence to 0 of the right hand side implies convergence to 0 of the left hand side. Thus we have a convergent subsequence of a Cauchy sequence, so the Cauchy sequence $\{z_n\}$ that we started with must as well be convergent with limit the element $x + Y$ in X/Y .