

The Baire category theorem (Version 1)

①

Let (X, d) be a complete metric space. If $\{V_n\}_{n=1}^{\infty}$ is a sequence of dense open sets, then $\bigcap_{n=1}^{\infty} V_n$ is dense in X .

Proof: We are given open sets $V_n, n \geq 1$, such that $\overline{V_n} = X$ for every $n \geq 1$. Recall that for a subset A of X to be dense in X it means that: $\forall W$ open, nonempty in X , we have $W \cap A \neq \emptyset$.

Set $G := \bigcap_{n=1}^{\infty} V_n$, let W be nonempty, open set. We must show:

(*) $W \cap G \neq \emptyset$. Let $B_r(x) = \{y \in X \mid d(y, x) < r\}$, open ball around x radius r .

Since $\overline{V_1} = X$, we have $W \cap V_1 \neq \emptyset$, and since $W \cap V_1$ is again open, there are $x_1 \in W \cap V_1$ and $r_1 > 0$ s.t. $B_{r_1}(x_1) \subset W \cap V_1$.

Choose $0 < r_1 < r_1$ s.t. $\overline{B_{r_1}(x_1)} \subset W \cap V_1$.

Since V_2 is dense (and open), we have nonempty (open) sets

$\underbrace{B_{r_1}(x_1) \cap V_2}_{\text{open}} \subset W \cap V_2$ and so there are $x_2 \in B_{r_1}(x_1) \cap V_2$

and $r_2 > 0$ s.t. $r_2 < \frac{r_1}{2}$ and $\overline{B_{r_2}(x_2)} \subset B_{r_1}(x_1) \cap V_2$.

Continuing this process, we find $\{x_n\}_{n \in \mathbb{N}} \subset X$ and $\{r_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ with

$$0 < r_n < \frac{r_{n-1}}{2} < \frac{r_1}{2^{n-1}} \quad \forall n \geq 2 \text{ s.t. } \overline{B_{r_n}(x_n)} \subset B_{r_{n-1}}(x_{n-1}) \cap V_n.$$

Thus $d(x_n, x_{n-1}) < r_{n-1} < \frac{r_1}{2^{n-2}} \rightarrow 0$ as $n \rightarrow \infty$. Therefore $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X , and since X is complete $\exists x = \lim_n x_n \in X$.

Now, for every $n \geq 1$, $x = \lim_{m \geq n} x_m \in \overline{B_{r_n}(x_n)} \subset V_n$, so $x \in G$.

Since $x \in \overline{B_{r_1}(x_1)} \subset W \cap V_1$, we have $x \in W$. In all, $x \in W \cap G$, as needed. $\square$

To state a second version of Baire category theorem, 21/10/2025
 we need some concepts. ②

Def: Let X be a topological space (for example, a metric space)

Given $A \subseteq X$, we define its

interior $A^\circ := \bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U$ and closure $\bar{A} = \bigcap_{\substack{A \subseteq C \\ C \text{ closed}}} C$.

We say that A is nowhere dense if $(\bar{A})^\circ = \emptyset$ (Norsh: intetsted tett)

Facts (see MAT 2400) Given $A \subseteq X$ (topological space, e.g. metric),

$$(a) X \setminus A^\circ = \overline{X \setminus A};$$

$$(b) (X \setminus A)^\circ = X \setminus \bar{A}.$$

Consequence: $A \subseteq X$ is nowhere dense if and only if $\overline{(X \setminus A)^\circ} = X$.

Proof: $(\bar{A})^\circ = \emptyset \Leftrightarrow X \setminus (\bar{A})^\circ = X \Leftrightarrow \overline{X \setminus \bar{A}} = X \Leftrightarrow \overline{(X \setminus A)^\circ} = X$.

Theorem (Baire category, meagre form) see [R, Thm 22.2].

A complete metric space is not the countable union of nowhere dense sets.

Proof: Let (X, d) be complete. Suppose that $\{F_n\}_{n=1}^\infty$ is a countable collection of nowhere dense subsets of X . We must show that $\bigcup_{n=1}^\infty F_n \neq X$. We will show $\bigcup_{n=1}^\infty \bar{F}_n \neq X$, which will imply

the claim (since $\bigcup_n F_n \subseteq \bigcup_n \bar{F}_n$).

$$\text{Now, } \bigcup_{n=1}^\infty \bar{F}_n \neq X \Leftrightarrow \emptyset \neq X \setminus \left(\bigcup_{n=1}^\infty \bar{F}_n \right) \Leftrightarrow \emptyset \neq \bigcap_{n=1}^\infty (X \setminus \bar{F}_n)$$

$$\Leftrightarrow \emptyset \neq \bigcap_{n=1}^\infty (X \setminus F_n)^\circ.$$

By assumption, each $(X \setminus F_n)^\circ$ is dense, $n \geq 1$, thus holds by version 1 of the theorem. $\square$

Consequence: If X is a complete metric space

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(3)

such that $X = \bigcup_{n=1}^{\infty} F_n$ where F_n is closed for all $n \geq 1$, then at least one of the F_n 's has nonempty interior (thus, contains an open ball).

Application: $X = \mathbb{N}$ with a metric d such that $\{n\}$ is closed but not open is not complete: $\overline{\{n\}} = \{n\}$, so $\{n\}^\circ = \emptyset$, but $X = \bigcup_n \{n\}$.

Sometimes the following terminology is used:

A subset $E \subseteq X$ in a topological space (for example, a metric space) is meager (or of the 1st category) if $E = \bigcup_{n=1}^{\infty} E_n$ where each E_n is nowhere dense. A set that is not of the 1st category is called of the 2nd category.

Baire category theorem (reformulated): A complete metric space is not a meager set. Equivalently, it is of the second category.

Applications of the Baire category theorem (also called residual, see [R]).

(I) Principle of uniform boundedness.

Theorem (UIB)

Let X be a Banach space and Y a normed space. Let $\mathcal{F} \subseteq B(X, Y)$ be a collection of bounded linear operators from X to Y such that

$$\sup_{T \in \mathcal{F}} \|Tx\|_Y < \infty \text{ for each } x \in X.$$

$$\text{Then, } \sup_{T \in \mathcal{F}} \|T\|_{B(X, Y)} < \infty.$$

Proof: For each $n \geq 1$, set $F_n := \{x \in X \mid \sup_{T \in \mathcal{F}} \|Tx\| \leq n\}$. Note that

F_n is closed, since $F_n = \bigcap_{T \in \mathcal{F}} \{x \in X \mid \|Tx\| \leq n\}$ and each $\{x \mid \|Tx\| \leq n\}$

is closed by continuity of $T \in B(X, Y)$. The hypothesis gives us that $X = \bigcup_{n=1}^{\infty} F_n$. Since X is complete, the Baire category theorem

implies that for some $n \geq 1$, $\exists x_0 \in X$ and $\delta > 0$ s.t. $\overline{B_{\delta}^X(x_0)} \subseteq F_n$.

(First, get an open ball inside F_n , then shrink radius to get the closure of an open ball inside F_n .)

Let $T \in \mathcal{F}$, let $x \in X$ with $\|x\| \leq 1$. Need $\|Tx\| < C$ ^{a constant} independent of T .

$$\|Tx\| = \|Tx \pm T(\delta^{-1}x_0)\| = \|T(x + \delta^{-1}x_0) - T(\delta^{-1}x_0)\|$$

$$\leq \|T(x + \delta^{-1}x_0)\| + \|T(\delta^{-1}x_0)\|$$

$$= \delta^{-1} \|T(\delta x + x_0)\| + \delta^{-1} \|Tx_0\|.$$

Now, $\|\delta x + x_0 - x_0\| = \|\delta x\| \leq \delta$, so $x_0, \delta x + x_0 \in \overline{B_{\delta}^X(x_0)}$. Thus

$\|T(\delta x + x_0)\| \leq n$ and $\|Tx_0\| \leq n$. In all, $\|Tx\| \leq 2n\delta^{-1}$, so

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\| \leq 2n\delta^{-1}. \text{ Hence } \sup_{T \in \mathcal{F}} \|T\| \leq 2n\delta^{-1} < \infty. \quad \square$$

Note: We have showed that for X, Y normed spaces, if the set

$\{x \in X \mid \sup_{T \in \mathcal{F}} \|Tx\| < \infty\}$ is not meager, then $\sup_{T \in \mathcal{F}} \|T\| < \infty$.

Corollary Let X be a Banach space, Y a normed space

and $\{T_n\}_{n \geq 1}$ a sequence of operators in $B(X, Y)$ s.t. $\{T_n x\}_{n \geq 1}$ converges

(in Y) for every $x \in X$. Then $Tx := \lim_{n \rightarrow \infty} T_n x$ for $x \in X$ determines an element $T \in B(X, Y)$.

(Warning: we are not asserting that $T_n \rightarrow T$ in $B(X, Y)$. This can fail).

Proof: Uniqueness of the limit implies that T given as above is linear. Given x , $\{T_n x\}_n$ is convergent hence it is bounded.

By PUB applied to $\mathcal{F} = \{T_n: X \rightarrow Y \mid n \geq 1\}$ we obtain

$$\begin{aligned} \sup_{n \geq 1} \|T_n\| < \infty. \text{ Therefore, } \|Tx\| &= \|\lim_{n \rightarrow \infty} T_n x\| = \lim_{n \rightarrow \infty} \|T_n x\| \leq \\ &\leq \left(\sup_{n \geq 1} \|T_n\| \right) \cdot \|x\|, \text{ so } T \text{ is bounded (with } \|T\| \leq \sup_{n \geq 1} \|T_n\| \text{)} \\ &\text{for every } x \in X \end{aligned}$$

Corollary (Condensation of singularities)

Given X Banach space, Y normed space, $\mathcal{F} \subseteq B(X, Y)$ with $\sup_{T \in \mathcal{F}} \|T\| = \infty$, there exists $x \in X$ s.t. $\sup_{T \in \mathcal{F}} \|Tx\| = \infty$.

Proof: Contrapositive.