

The Baire Category Theorem

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Main application: Existence of a continuous 2π -periodic function whose Fourier series at 0 does not converge.

Setup: $X = \{f \in C([- \pi, \pi]) \mid f(-\pi) = f(\pi)\}$, space of 2π periodic functions, with norm $\|f\|_\infty = \max_{t \in [-\pi, \pi]} |f(t)|$. Then X is a closed subspace of $C([- \pi, \pi])$, so is a Banach space.

Need: $\{e_k\}_{k \in \mathbb{Z}}$, $e_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$, $k \in \mathbb{Z}$, $e_k \in X$ and $e_k \in L^2[-\pi, \pi]$

For $f \in L^2[-\pi, \pi]$, its n -Fourier coefficient is $\frac{1}{\sqrt{2\pi}} \langle f, e_n \rangle$, so

$$c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) \overline{e_n(t)} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt.$$

The n 'th partial sum is $f_n = \sum_{k=-n}^n \langle f, e_k \rangle e_k \in X$, so

$$f_n(x) = \sum_{k=-n}^n \left(\int_{-\pi}^{\pi} f(t) \frac{e^{-ikt}}{\sqrt{2\pi}} dt \right) \frac{e^{ikx}}{\sqrt{2\pi}} = \frac{1}{2\pi} \sum_{k=-n}^n \left(\int_{-\pi}^{\pi} f(t) e^{-ikt} dt \right) e^{ikx}.$$

We are interested in $f_n(0) = \frac{1}{2\pi} \sum_{k=-n}^n \left(\int_{-\pi}^{\pi} f(t) e^{-ikt} dt \right)$.
(For simplicity, we use $x=0$).

Note that $f_n(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left(\sum_{k=-n}^n e^{-ikt} \right) dt$.

Let $K_n(t) = \sum_{k=-n}^n e^{-ikt}$, $n \geq 1$, the Dirichlet kernel. Then

$$K_n(t) = \sum_{k=-n}^n e^{ikt} = e^{-int} \sum_{k=0}^{2n} (e^{it})^k = \begin{cases} e^{-int} \cdot \frac{e^{it(2n+1)} - 1}{e^{it} - 1}, & \text{if } t \notin 2\pi\mathbb{Z} \\ 2n+1, & \text{if } t \in 2\pi\mathbb{Z} \end{cases}$$

By de Moivre's formula, $K_n(t) = \frac{\sin(n + \frac{1}{2})t}{\sin \frac{t}{2}}$, $\forall t \neq 2\pi\mathbb{Z}$.

Also, $K_n(0) = K_n(2\pi) = 2n+1$.

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Claim The family of linear operators $\mathcal{F} = \{S_n : n \in \mathbb{N}\}$ where $S_n : X \rightarrow \mathbb{R}$, $S_n(f) = f_n(0)$ is in $B(X, \mathbb{R})$ and is not uniformly bounded, meaning $\sup_n \|S_n\| = \infty$.

Consequence: By condensation of singularities, $\exists f \in X$ so that $\sup_n \|S_n f\| = \infty$. In other words, $\exists f \in X$ s.t. $\sup_n |f_n(0)| = \infty$. This means that $\{f_n(0)\}_n$ cannot converge, so the Fourier series of f at 0 does not converge.

Proof of Claim:

(I) $S_n \in B(X, \mathbb{R})$

(II) $\sup \{ |S_n f| : f \in X, \|f\|_\infty \leq 1 \} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |K_n(t)| dt = \frac{1}{2\pi} \|K_n\|_1$

(III) $\|K_n\|_1 > \frac{8}{\pi} \sum_{k=1}^n \frac{1}{k} \rightarrow \infty$ as $n \rightarrow \infty$.

(I) $|S_n(f)| = |f_n(0)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) K_n(t) dt \right| \leq \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} |K_n(t)| dt \right) \cdot \|f\|_\infty$

Thus $S_n \in B(X, \mathbb{R})$ with $\|S_n\| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |K_n(t)| dt$.

(II) Need to show $\geq$. The idea is to replace $|K_n(t)|$ with the function $K(t) \cdot \text{sign}(K(t))$. Issue: this is not continuous, so we need an approximation by continuous functions. We do this more generally. (See page 7).

Observation If we let $\mathcal{F} = \{T_n : n \geq 1\}$ where $T_n : X \rightarrow X$ is given by $T_n f = f_n$, then it also holds that

$\sup_{n \geq 1} \|T_n\| = \infty$, thus $\exists f$ s.t. $\sup_{n \geq 1} \|T_n f\|_\infty = \infty$, meaning that

the Fourier series of f is not uniformly convergent.

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$$\|K_n\|_1 = \int_{-\pi}^{\pi} |K_n(t)| dt = \int_{-\pi}^{\pi} \frac{|\sin(n+\frac{1}{2})t|}{|\sin \frac{t}{2}|} dt \geq \text{use } |\sin u| \leq |u| \quad \forall u$$

$$\geq \int_{-\pi}^{\pi} \frac{|\sin(n+\frac{1}{2})t|}{|\frac{t}{2}|} dt = 2 \int_0^{\pi} \frac{|\sin(n+\frac{1}{2})t|}{|\frac{t}{2}|} dt$$

Use that $\frac{n\pi}{n+\frac{1}{2}} < \pi$ to bound this below. (We know that $\int_0^{\pi} |\sin t| dt = 2$ etc on $[\pi, 2\pi]$, $2\pi, 3\pi$, $[(k-1)\pi, k\pi]$, ..., $[(n-1)\pi, n\pi]$.)

$$> 2 \int_0^{\frac{n\pi}{n+\frac{1}{2}}} \frac{|\sin(n+\frac{1}{2})t|}{|\frac{t}{2}|} dt$$

Split up over subintervals $[(k-1)\pi/(n+\frac{1}{2}), k\pi/(n+\frac{1}{2})]$, $k=1, \dots, n$.

$$= 2 \sum_{k=1}^n \int_{\frac{(k-1)\pi}{n+\frac{1}{2}}}^{\frac{k\pi}{n+\frac{1}{2}}} \frac{|\sin(n+\frac{1}{2})t|}{|\frac{t}{2}|} dt. \quad \text{Estimate } t \text{ with upper bound of each subinterval.}$$

$$> 4 \sum_{k=1}^n \int_{\frac{(k-1)\pi}{n+\frac{1}{2}}}^{\frac{k\pi}{n+\frac{1}{2}}} \frac{|\sin(n+\frac{1}{2})t|}{k\pi} (n+\frac{1}{2}) dt : \text{change of variable}$$

$$= 4 \sum_{k=1}^n \frac{1}{k\pi} \left(\int_{(k-1)\pi}^{k\pi} |\sin y| dy \right) = 2$$

$y = (n+\frac{1}{2})t$
 $dy = (n+\frac{1}{2})dt$
on $(k-1)\pi$ to $k\pi$

$$= \frac{8}{\pi} \sum_{k=1}^n \frac{1}{k}$$

$$\text{So } \|S_n\| = \frac{1}{2\pi} \|K_n\|_1 > \frac{4}{\pi^2} \sum_{k=1}^n \frac{1}{k} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

This proves our claim. Hence the consequence gives what we wanted.

General fact. If $\Phi: [a, b] \rightarrow \mathbb{K}$ is cont., then $T_\Phi: C[a, b] \rightarrow \mathbb{K}$ given by $T_\Phi(f) = \int_a^b \Phi(y)f(y)dy$ is bounded with $\|T_\Phi\| = \|\Phi\|_1$.

Proof: That $\|T_\Phi f\|_\infty \leq \|\Phi\|_1 \|f\|_\infty$ is easy to check.

Need to show $\geq$. For this, we need to reverse the ineq.

$$|\int \Phi(t)f(t)dt| \leq (\int |\Phi(t)|dt) \|f\|_\infty.$$

At each t , we can use $\alpha = \frac{\overline{\Phi(t)}}{|\Phi(t)|}$ when $\Phi(t) \neq 0$, with $|\alpha| = 1$.

But this does not give a cont. function. We get the sign function

$$\text{sign}(\overline{\Phi})(t) = \begin{cases} \frac{\overline{\Phi(t)}}{|\Phi(t)|} & \text{if } \Phi(t) \neq 0 \\ 0 & \text{if } \Phi(t) = 0 \end{cases} \quad \text{with values } +1 \& -1 \text{ everywhere}$$

where $\Phi(t) \neq 0$. Need an approximation by continuous functions.

For $\varepsilon > 0$, let $f_\varepsilon(t) = \frac{\overline{\Phi(t)}}{\sqrt{\Phi(t)^2 + \varepsilon}}$, then $f_\varepsilon(t) \in C[a, b]$, $\|f_\varepsilon\| \leq 1$,

for each $\varepsilon > 0$, and $\lim_{\varepsilon \rightarrow 0} f_\varepsilon(t) = \text{sign}(\overline{\Phi})(t)$. Pick $\{\varepsilon_n\}_n \searrow 0$.

$$\text{Then } \|\Phi\|_1 = \int_a^b |\Phi(t)|dt = \int_a^b \Phi(t) \text{sign}(\overline{\Phi})(t)dt = \int_a^b \lim_{n \rightarrow \infty} f_{\varepsilon_n}(t) \Phi(t)dt$$

$$\stackrel{\text{DCT}}{=} \lim_{n \rightarrow \infty} \int_a^b f_{\varepsilon_n}(t) \Phi(t)dt = \lim_{n \rightarrow \infty} \left| \int_a^b f_{\varepsilon_n}(t) \Phi(t)dt \right|,$$

$$= \lim_{n \rightarrow \infty} |T_\Phi(f_{\varepsilon_n})| \leq \|T_\Phi\| \text{ since } \|f_{\varepsilon_n}\|_\infty \leq 1, \forall n.$$

This proves the claim $\|T_\Phi\| = \|\Phi\|_1$.