

The Open Mapping Theorem

Let X and Y be Banach spaces and $T \in B(X, Y)$. If T is surjective, then T is an open mapping, in the sense that $T(O)$ is open in Y for every open set O in X .

Proof: For $\varepsilon > 0$, let $B_\varepsilon^X(x) = \{x' \in X \mid \|x - x'\|_X < \varepsilon\}$ open ball centered at x in X with radius ε and $B_\varepsilon^Y(y) = \{y' \in Y \mid \|y - y'\|_Y < \varepsilon\}$ similar open ball in Y .

Let $O \subseteq X$ be open. Let $x \in O$. There exists $r > 0$ s.t. $B_r^X(x) \subseteq O$.

We have $T(x) \in T(O)$. Need to find $\varepsilon > 0$ s.t. $B_\varepsilon^Y(Tx) \subseteq T(O)$.

Will show: $\boxed{\exists \varepsilon > 0 \text{ s.t. } B_\varepsilon^Y(Tx) \subseteq T(B_r^X(x))}$. (1)

By hypothesis, $T(X) = Y$. Since $X = \bigcup_{n=1}^{\infty} B_n^X(0)$, $T(X) = \bigcup_{n=1}^{\infty} T(B_n^X(0))$,

so $Y = \bigcup_{n=1}^{\infty} T(B_n^X(0))$. Since Y is complete, Baire's category

Theorem implies that $\exists n \geq 1$ s.t. $\overline{T(B_n^X(0))}^\circ \neq \emptyset$. Thus,

$\overline{T(B_n^X(0))}$ contains some open set, which can be assumed of

the form $B_{\varepsilon_1}^Y(y)$, for some $y \in Y$, $\varepsilon_1 > 0$.

Next, use linearity to show: $\boxed{(2) \ B_{\varepsilon_1}^Y(0) \subseteq \overline{T(B_{2n}^X(0))}}$. Let $\varepsilon_2 > 0$.

$y \in B_{\varepsilon_1}^Y(y) \subseteq \overline{T(B_n^X(0))} \Rightarrow \exists x \in B_n^X(0)$ s.t. $\|y - Tx\| < \frac{\varepsilon_2}{2}$.

Note: $y + B_{\varepsilon_1}^Y(0) = \{y + y' \mid \|y'\| < \varepsilon_1\} \subseteq B_{\varepsilon_1}^Y(y) \subseteq \overline{T(B_n^X(0))}$, so
since $\|y + y' - y\| < \varepsilon_1$

$\exists x_2 \in B_n^X(0)$ s.t. $\|y + y' - Tx_2\| < \frac{\varepsilon_2}{2}$. Then $\|x_2 - x\| \leq n + n = 2n$

and $\|T(x_2 - x) - y'\| = \|Tx_2 - (y' + y) + y - Tx\| < \frac{\varepsilon_2}{2} + \frac{\varepsilon_2}{2} = \varepsilon_2$.

Since ε_2 is arbitrary, we get $y' \in T(B_{2n}^X(0))$. But $y' \in B_{\varepsilon_1}^Y(0)$ is arbitrary, so (2) follows.

Now, we claim that ~~$T(az) = aTz$ for all $a \in K$ and $z \in X$ to prove~~

$$(3) \forall \alpha > 0, \overline{B_{\frac{\varepsilon_1}{2n}}^Y(0)} \subseteq T(\overline{B_\alpha^X(0)}).$$

The map $f_a: B_{\varepsilon_1}^Y(0) \rightarrow B_{a\varepsilon_1}^Y(0)$, $f(z) = az$, for $a \in K$, is surjective.

(If $u \in B_{a\varepsilon_1}^Y(0)$, then $u = f(a^{-1}u)$ and $\|a^{-1}u\| < \varepsilon_1$).

We also have $f_a: T(B_{2n}^X(0)) \rightarrow T(B_{2n \cdot a}^X(0))$ surjective.

(If $u = Tx_1 \in T(B_{2n \cdot a}^X(0))$, then $u = a \cdot (a^{-1}u) = f_a(a^{-1}u)$ with
 $a^{-1}u = a^{-1}(Tx_1) = T(a^{-1}x_1) \in T(B_{2n}^X(0))$).

Since f_a is a homeomorphism, it is also surjective from $\overline{T(B_{2n}^X(0))}$ onto $\overline{T(B_{2n \cdot a}^X(0))}$, for arbitrary $\alpha > 0$.

Now let $\alpha > 0$ and set $a = \frac{\alpha}{2n}$. Applying f_a to both sets in (2),

we get $\overline{B_{\frac{\alpha}{2n} \cdot \varepsilon_1}^Y(0)} \subseteq \overline{T(B_\alpha^X(0))}$, as claimed in (3).

Next, we prove (4): $\overline{B_{\frac{\varepsilon_1}{2n}}^Y(0)} \subseteq T(\overline{B_1^X(0)})$. Denote $\delta = \frac{\varepsilon_1}{2n} > 0$.

Let $y \in B_\delta^Y(0)$, and pick $\alpha_1 \in (\frac{\|y\|}{\delta}, 1)$. Choose $(\alpha_n)_{n \geq 2}$ positive reals such that $\sum_{n=1}^{\infty} \alpha_n < 1$ (for example, a geometric series).

By (3) applied to α_1 we get $y \in \overline{T(B_{\alpha_1}^X(0))}$.

Hence $\exists x_1 \in B_{\alpha_1}^X(0)$ s.t. $\|y - Tx_1\| < \alpha_2 \delta$.

By (3) applied to α_2 we get $y - Tx_1 \in \overline{T(B_{\alpha_2}^X(0))}$.

Hence $\exists x_2 \in B_{\alpha_2}^X(0)$ s.t. $\|y - Tx_1 - Tx_2\| < \alpha_3 \delta$. Continuing we find $x_k \in B_{\alpha_k}^X(0)$ s.t. $\|y - T(x_1 + x_2 + \dots + x_n)\| < \alpha_{n+1} \delta$, for all $n \geq 1$.

Since $\sum_{k=1}^{\infty} \|x_k\| < \sum_{k=1}^{\infty} \alpha_k < 1$, the series $\sum_{k=1}^{\infty} x_k$ is convergent in X , because X is a Banach space. Let $x_0 := \sum_{k=1}^{\infty} x_k \in X$. Note that

$$\|x\| < 1. \text{ In all, } y = \lim_{n \rightarrow \infty} \sum_{k=1}^n T(x_k) = \sum_{k=1}^{\infty} T(x_k) = T x_0 \text{ by}$$

continuity of T . Since y was arbitrary in $B_{\delta}^Y(0)$ and $x_0 \in B_1^X(0)$, we have proved (4).

Finally, we are ready to prove (1): recall that $B_r^X(x) \subseteq \mathcal{O}$.

From $\|x' - x\| < r$ we write $x' = x + (x' - x) = x + r(\underbrace{r^{-1}(x' - x)})$, and see that $B_r^X(x) = x + r B_1^X(0)$. Similarly in Y . $\in B_1^X(0)$

With $\delta = \frac{\varepsilon}{2n}$ from (2), we have now

$$B_{r\delta}^Y(Tx) = Tx + r\delta B_1^Y(0)$$

$$= Tx + r B_{\delta}^Y(0) \quad (\text{Verify})$$

$$\stackrel{(4)}{\subseteq} Tx + r T(B_1^X(0))$$

$$= T(x + B_1^X(0)) \text{ by linearity}$$

$$= T(B_r^X(x))$$

$$\subseteq T(\mathcal{O}), \text{ as needed for (1).} \quad \square$$