

MAT4410 Lecture 21

Applications of The Open Mapping Theorem

Recall: [Open Mapping Theorem] If X, Y are Banach spaces and $T \in B(X, Y)$ satisfies $T(X) = Y$ (meaning T is surjective), then T is an open mapping (meaning if $V \subseteq X$ open $\Rightarrow T(V) \subseteq Y$ open).

(I) The Bounded Inverse Theorem

Let X and Y be Banach spaces. If $T \in B(X, Y)$ and T is bijective, then the inverse map $T^{-1}: Y \rightarrow X$ is in $B(Y, X)$. Furthermore, $\|T^{-1}\| \geq \frac{1}{\|T\|}$. We say that the bounded operator T is invertible.

Proof: First, to see that T^{-1} is linear, let $\alpha_1, \alpha_2 \in \mathbb{C}$ and $y_1, y_2 \in Y$. Since T is bijective, there are unique $x_1, x_2 \in X$ s.t. $T(x_j) = y_j, j=1, 2$. Then $y_1 + y_2 = T(x_1) + T(x_2) = T(x_1 + x_2)$, so $x_1 + x_2 = T^{-1}(y_1 + y_2) = T^{-1}(y_1) + T^{-1}(y_2)$. Similarly, $T^{-1}(\alpha y) = \alpha T^{-1}(y), \forall y \in Y, \alpha \in \mathbb{C}$.

To show that T is bounded, it suffices to show T^{-1} is continuous.

For $\mathcal{O} \subseteq X$ open, we have $(T^{-1})^{-1}(\mathcal{O}) = T(\mathcal{O})$, which is open in Y by the Open Mapping Theorem. This property is equivalent to T^{-1} being continuous.

For $y \in Y, \|y\| = \|T(T^{-1}(y))\| \leq \|T\| \cdot \|T^{-1}y\| \leq \|T\| \cdot \|T^{-1}\| \cdot \|y\|$, so

$1 \leq \|T\| \cdot \|T^{-1}\|$, from which we get the inequality in the theorem.

→ Alternative proof: Recall from the Open Mapping Theorem $\square$

that $\exists \delta > 0$ s.t. $\delta B_1^Y(0) \subseteq T(B_1^X(0))$. Since T is bijective, its inverse, T^{-1} , satisfies $T^{-1}(\delta B_1^Y(0)) \subseteq B_1^X(0)$, so

$T^{-1}(B_1^Y(0)) \subseteq \frac{1}{\delta} B_1^X(0)$. This means, that $\|T^{-1}\| \leq \frac{1}{\delta} < \infty$, so T is bounded. $\square$

(II) The Closed Graph Theorem

(Preparation: For $T: X \rightarrow Y$ linear map between normed spaces X, Y , the graph of T is the set

$$G(T) = \{(x, Tx) \mid x \in X\}. \text{ Note that } G(T) \subseteq X \times Y.$$

Given X, Y , two normed spaces, we can form their direct sum:

Standard notation (not orthogonality) $X \oplus Y = \{(x, y) \mid (x, y) \in X \times Y\}$ with $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
 $\alpha \cdot (x_1, y_1) = (\alpha x_1, \alpha y_1), \alpha \in \mathbb{K}$
 $x_1, x_2 \in X$
 $y_1, y_2 \in Y$

Fact: $X \oplus Y$ becomes a normed space with

(i) $\|(x, y)\| := \|x\|_X + \|y\|_Y$;
 (Verify that this is a norm).

Exercise (ii) $\|(x, y)\|_{\max} := \max(\|x\|_X, \|y\|_Y)$. In fact, the two norms are equivalent. If $T: X \rightarrow Y$ is linear, then $G(T)$ is a subspace of $X \oplus Y$.

Proposition If X and Y are Banach spaces, then $X \oplus Y$ is a Banach space. (Proof: Routine exercise).

Definition: A linear operator $T: X \rightarrow Y$ between normed spaces is said to be closed if $G(T)$ is a closed subspace of $X \oplus Y$.

(Theorem (Closed Graph))

Let X and Y be Banach spaces and $T: X \rightarrow Y$ a linear operator. Then T is bounded if and only if it is closed.

Proof: Assume T is closed. Then $G(T)$ is a Banach space.

Define the projection maps $\pi_1: G(T) \rightarrow X, \pi_1(x, Tx) = x, (x, Tx) \in G(T)$ and $\pi_2: G(T) \rightarrow Y, \pi_2(x, Tx) = Tx, (x, Tx) \in G(T)$. Clearly linear.

From $\|\pi_1(x, Tx)\| = \|x\| \leq \|(x, Tx)\|_{\max}$, get $\|\pi_1\| \leq 1$,

and similarly $\|\pi_2\| \leq 1$. Now, π_1 is clearly surjective and injective, so $\pi_1^{-1} \in B(X, G(T))$. For every $x \in X$, $\pi_2(\pi_1^{-1}(x)) = \pi_2(x, Tx) = Tx$, so $T = \pi_2 \circ \pi_1^{-1}$ and is bounded as the composition of bounded linear operators.

The other implication is a routine verification: suppose that T is bounded, thus continuous.

Let $\{w_n\}_n \in G(T)$ be convergent to some $w \in X \oplus Y$.

We have $w_n = (x_n, Tx_n)$ for some sequence $\{x_n\}_n \in X$ and $w = (x, y)$. Since π_1 is continuous, $\lim_n x_n = \lim_n \pi_1(x_n, Tx_n) = \pi_1(w) = x$.

Since T is continuous, we have $Tx = T(\lim_n x_n) = \lim_n Tx_n$, and

$$\text{so } Tx = \lim_n \pi_2(x_n, Tx_n) = \pi_2(\lim_n (x_n, Tx_n)) = \pi_2(\lim_n w_n) = \pi_2(w) = y.$$

In all, $w = (x, Tx) \in G(T)$, which shows precisely that $G(T)$ is closed. $\square$

Note: In the proof, we established the following for $T: X \rightarrow Y$ linear, X, Y Banach sp.

T is closed $\Leftrightarrow$ for all convergent sequences $\{x_n\}_n \in X$ such that $\{Tx_n\}_n$ is convergent in Y , we have $\lim_n Tx_n = T(\lim_n x_n)$.

Compare with:

T bounded $\Leftrightarrow$ for all convergent sequences $\{x_n\}_n \in X$, the sequence $\{Tx_n\}_n$ is convergent and $\lim_n Tx_n = T(\lim_n x_n)$.

Point: It is easier to verify that an operator is closed than to show directly that ~~it is~~ bounded: we moved existence of the limit $\lim_n Tx_n$ to the hypothesis. ~~this is easier~~

Example: Assume $T: \mathcal{H} \rightarrow \mathcal{H}$ is linear (for $\mathcal{H}$ Hilbert space) and there exists $T^*: \mathcal{H} \rightarrow \mathcal{H}$ linear such that $\langle Tx, y \rangle = \langle x, T^*y \rangle, \forall x, y \in \mathcal{H}$. Then $T \in B(\mathcal{H})$.

Proof: We show that T is closed. Assume $x_n \rightarrow x \in \mathcal{H}$ and $Tx_n \rightarrow y \in \mathcal{H}$.

$$\text{For any } z \in \mathcal{H}, \langle Tx, z \rangle = \langle x, T^*z \rangle = \lim_n \langle x_n, T^*z \rangle = \lim_n \langle Tx_n, z \rangle = \langle y, z \rangle, \text{ so } Tx - y \in \mathcal{H}^\perp, \text{ hence } Tx = y.$$

Thus, $G(T)$ is closed and the claim follows.

Consequences of the Bounded Inverse theorem

① If X is a Banach space under two norms and these are comparable, then they are equivalent.

Proof: Suppose $\|\cdot\|_1, \|\cdot\|_2$ are two norms on X such that $\exists C > 0$ with $\|x\|_1 \leq C\|x\|_2$, for every $x \in X$.

Consider the identity linear operator $I: (X, \|\cdot\|_2) \rightarrow (X, \|\cdot\|_1)$.

The assumption means that $I \in B((X, \|\cdot\|_2), (X, \|\cdot\|_1))$. Since I is surjective, by the Bounded Inverse theorem it will be invertible.

So $\exists C' > 0$ s.t. $\|x\|_2 \leq C'\|x\|_1, \forall x \in X$.

② For X complex Banach space, we defined the spectrum of $T \in B(X)$ as

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not invertible}\}.$$

Let $\rho(T) = \mathbb{C} \setminus \sigma(T)$, the resolvent set of T . Thus,

$$\rho(T) = \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is invertible}\}.$$

By the Bounded Inverse theorem, we have the useful rephrasing:

$$\rho(T) = \{\lambda \in \mathbb{C} \mid \lambda I - T: X \rightarrow X \text{ is bijective}\}.$$

③ Schauder bases in separable Banach spaces (Without proofs.) See [R, 23.2],

Definition A sequence $\{e_n\}_{n=1}^{\infty}$ in a separable Banach space is a Schauder basis for X if every $x \in X$ has a unique expression

$$x = \sum_{j=1}^{\infty} \alpha_j e_j \text{ for some } \alpha_j \in K, \text{ with the sum converging in } X.$$

Example: $\{e_j\}_{j=1}^{\infty}$ in $\ell^p, 1 \leq p < \infty$. Denote $P_n x = \sum_{j=1}^n \alpha_j e_j$, for $n \geq 1$.

Fact/Prop: If $\{e_j\}_{j=1}^{\infty}$ is a Schauder basis for X sep. with $\|e_j\| = 1, \forall j$,

Analogous of Bessel's inequality for $X = \ell_2$, Hilbert space.
 Then $\|x\| = \sup_n \left\| \sum_{j=1}^n \alpha_j e_j \right\|_X$ is a norm and $\exists C > 0$ s.t. $\|x\| \leq C\|x\|_X, \forall x$.
 In particular, $\|\cdot\|$ & $\|\cdot\|_X$ are equivalent.

(20.4) Adjoint of linear maps between Banach spaces

Let X, Y Banach spaces and $T \in B(X, Y)$.

Let $Y^* = B(Y, \mathbb{K})$, and $X^* = B(X, \mathbb{K})$. For $g \in Y^*$, form $g \circ T$.

$$X \xrightarrow{T} Y \xrightarrow{g} \mathbb{K}$$

Fact ① $g \circ T \in X^*$ with $\|g \circ T\|_{X^*} \leq \|T\| \cdot \|g\|_{Y^*}$ (Routine)

The adjoint of T is defined to be the map $T^*: Y^* \rightarrow X^*$ given by

$$T^*g = g \circ T, \quad \forall g \in Y^*.$$

Fact ② T^* is linear (Routine).

From fact ①, we get $T^* \in B(Y^*, X^*)$ with $\|T^*\| \leq \|T\|$.

Claim: equality holds.

Let $x \in X$. Need $\|Tx\| \leq \|T^*\| \cdot \|x\|$. Since $\|T^*\|$ is computed on $\|T^*g\|$, we need an element g in Y^* .

Assume $\|x\| \leq 1$. By the consequence to Hahn-Banach theorem,

$\exists g \in Y^*$ with $\|g\| = 1$ and $g(Tx) = \|Tx\|$. Then

$$\|Tx\| = g(Tx) = (T^*g)(x) = |T^*g(x)| \leq \|T^*g\| \cdot \|x\| \leq \|T^*\| \cdot \|g\| \cdot \|x\|,$$

so $\|Tx\| \leq \|T^*\|$, from which we conclude $\|T\| = \|T^*\|$.

Lemma If $T: X \rightarrow Y$ is an isomorphism of Banach spaces, then $T^*: Y^* \rightarrow X^*$ is an isomorphism. If T is isometric, so is T^* .

Proof: Routine verification. Show that T^* is invertible by showing that its inverse is $(T^{-1})^*$, see diagrams of maps.

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \uparrow T^{-1} & & \uparrow (T^{-1})^* \\ X^* & \xrightarrow{(T^{-1})^*} & Y^* \end{array} \quad \begin{aligned} g \in Y^* & \xrightarrow{T^*} X^* \\ ((T^{-1})^* \circ T^*)(g) &= (T^{-1})^*(g \circ T) \\ &= (g \circ T) \circ T^{-1} = g \circ (T \circ T^{-1}) \\ &= g \circ I_Y = g, \quad \forall g \in Y^*. \end{aligned}$$