

(20.4) Adjoint of linear maps between Banach spaces

Let X, Y Banach spaces and $T \in B(X, Y)$.

Let $Y^* = B(Y, \mathbb{K})$, and $X^* = B(X, \mathbb{K})$. For $g \in Y^*$, form $g \circ T$.

$$X \xrightarrow{T} Y \xrightarrow{g} \mathbb{K}$$

Fact ① $g \circ T \in X^*$ with $\|g \circ T\|_{X^*} \leq \|T\| \cdot \|g\|_{Y^*}$ (Routine)

The adjoint of T is defined to be the map $T^*: Y^* \rightarrow X^*$ given by

$$T^*g = g \circ T, \quad \forall g \in Y^*.$$

Fact ② T^* is linear (Routine).

From fact ①, we get $T^* \in B(Y^*, X^*)$ with $\|T^*\| \leq \|T\|$.

Claim: equality holds.

Let $x \in X$. Need $\|Tx\| \leq \|T^*\| \cdot \|x\|$. Since $\|T^*\|$ is computed on $\|T^*g\|$, we need an element g in Y^* .

Assume $\|x\| \leq 1$. By the consequence to Hahn-Banach theorem, $\exists g \in Y^*$ with $\|g\| = 1$ and $g(Tx) = \|Tx\|$. Then

$$\|Tx\| = g(Tx) = (T^*g)(x) = |T^*g(x)| \leq \|T^*g\| \cdot \|x\| \leq \|T^*\| \cdot \|g\| \cdot \|x\|,$$

so $\|Tx\| \leq \|T^*\|$, from which we conclude $\|T\| = \|T^*\|$.

Lemma If $T: X \rightarrow Y$ is an isomorphism of Banach spaces, then $T^*: Y^* \rightarrow X^*$ is an isomorphism. If T is isometric, so is T^* .

Proof: Routine verification. Show that T^* is invertible by showing that its inverse is $(T^{-1})^*$, see diagrams of maps.

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \uparrow T^{-1} & & \uparrow (T^{-1})^* \\ X^* & \xrightarrow{T^*} & Y^* \end{array} \quad \begin{aligned} ((T^{-1})^* \circ T^*)(g) &= (T^{-1})^*(g \circ T) \\ &= (g \circ T) \circ T^{-1} = g \circ (T \circ T^{-1}) \\ &= g \circ I_X = g, \quad \forall g \in Y^*. \end{aligned}$$

The second dual

For X a normed space over $\mathbb{K}$, we know that X^* is again a normed space, even a Banach space. We form the dual of X^* , denoted by X^{**} , so

$$X^{**} = B(X^*, \mathbb{K})$$

Observe that: if $x \in X$, then we can define an element $\psi \in X^{**}$ by $\psi(f) = f(x)$, for each $f \in X^*$.

Fact: this ψ "depends on x ". We denote it x^{**} (or $J(x)$).

Lemma (Important).

The map $J: X \rightarrow X^{**}$ which sends x to x^{**} as above is a linear and isometric map. In particular, it is injective and maps onto a subspace of X^{**} , which is closed if X is a Banach space.

Proof: J is well-defined. ^{Let $x \in X$.} Need x^{**} to be bounded. For $f \in X^*$ with $\|f\| \leq 1$, we have $\|x^{**}(f)\| = |f(x)| \leq \|f\| \cdot \|x\| \leq \|x\|$, so $J(x) = x^{**} \in B(X^*, \mathbb{K})$ with $\|J(x)\| \leq \|x\|$ for every x .

We have equality if we let $f \in X^*$ be the support functional of $x \neq 0$ in X : $|x^{**}(f)| = |f(x)| = \|x\| = \|f\| \cdot \|x\| \leq \|f\| \cdot \|x^{**}\|$.

Thus $\|J(x)\| = \|x\|$ for every x , so J is isometric.

J linear (routine): $J(x+y)(f) = f(x+y) = f(x) + f(y) = J(x) + J(y)$, $\forall x, y \in X$.

If X is a Banach space, assume $\{\psi_n\}_n \subseteq J(X)$ converges to some $\psi \in X^{**}$. We need to show that $\psi = x^{**}$ for some $x \in X$.

For each $n \geq 1$, we can write $\psi_n = J(x_n)$, $x_n \in X$, with $\|J(x_n)\| = \|x_n\|$.

For n, m , $\|\psi_n - \psi_m\| = \|J(x_n) - J(x_m)\| = \|J(x_n - x_m)\| = \|x_n - x_m\|$, so

$\{x_n\}_n \subseteq X$ is Cauchy, thus convergent. Let $x = \lim x_n$. Then $\|\psi_n - x^{**}\| = \|J(x_n - x)\| = \|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$, so $x^{**} = \lim \psi_n = \psi$. $\square$

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Call $J: X \rightarrow X^{**}$ the canonical embedding.

Note: since $\|J(x)\| = \|x\|$, $\forall x \in X$, we have $\ker J = \{0\}$, so J injective.

Definition: A Banach space X is reflexive if J is surjective, that is, if $J(X) = X^{**}$.

So, for X reflexive, the canonical embedding $J: X \rightarrow X^{**}$ is an isometric isomorphism.

Warning It is possible to have an isometric isomorphism $T: X \rightarrow X^{**}$ without X being reflexive. ($T \neq J$)

Examples of reflexive spaces

1) X finite dimensional.

claim ① $\dim X^* \leq \dim X$. (Hint: let $n = \dim X$, let $\{e_1, \dots, e_n\}$ a basis. Consider the map $X^* \rightarrow \mathbb{K}^n$, $\varphi \mapsto (\varphi(e_1), \dots, \varphi(e_n))$)

claim ② $\dim X^* \geq \dim X$. (Hint: Hahn-Banach).

From $\dim X = \dim X^*$ deduce: $\dim X = \dim X^{**}$, so J surjective.

2) $X = L^p(\Omega)$ for $1 < p < \infty$.

Let q be the dual exponent, so $\frac{1}{p} + \frac{1}{q} = 1$. Recall the isometric isomorphisms: (MAT 3400/4400)

$$\Psi_p: L^q(\Omega) \rightarrow (L^p(\Omega))^*, \quad \Psi_p(g)(f) = \int_{\Omega} fg \, d\mu, \quad g \in L^q, f \in L^p.$$

$$\Psi_q: L^p(\Omega) \rightarrow (L^q(\Omega))^*, \quad \Psi_q(f)(g) = \int_{\Omega} fg \, d\mu.$$

Take $\varphi \in (L^p(\Omega)^*)^*$. Need $\varphi = J(f)$ for some $f \in L^p(\Omega)$.

So, need $f \in L^p(\Omega)$ s.t. $\varphi(\Psi) = J(f)(\Psi) = \Psi(f)$ for all $\Psi \in L^p(\Omega)^*$.

Now, note that $\varphi \circ \psi_p: L^2(\Omega) \rightarrow \mathbb{C}$ is in $(L^2(\Omega))^*$, so it is in the image of $L^p(\Omega)$ under ψ_q . Therefore there is a unique $f \in L^p(\Omega)$ with

$$(\#) \quad \psi_q(f) = \varphi \circ \psi_p.$$

We use this f : let $\psi \in L^p(\Omega)^*$, so of form $\psi = \psi_p(g)$ for some $g \in L^q(\Omega)$. Then

$$\psi(\psi) = \psi(\psi_p(g)) = (\varphi \circ \psi_p)(g) \stackrel{(\#)}{=} \psi_q(f)(g)$$

$$= \int_{\Omega} g f d\mu$$

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$$= \psi_p(g)(f) = \psi(f) \text{ by choice of } \psi. \text{ Our claim is}$$

proved.

Alternatively, we know from earlier result that ψ_p^* is an isometric linear isomorphism, so also $(\psi_p^*)^{-1}: L^2(\Omega)^* \rightarrow L^p(\Omega)^{**}$ is an isomorphism. Verify that $J = (\psi_p^*)^{-1} \circ \psi_q$, thus J is surjective as a composition of surjective maps.

Proposition If X is a separable and reflexive Banach space, then X^{**} is separable.

Proof: Verify that if $\{x_n\}_n$ is dense in X , then $\{J(x_n)\}_n$ is dense in X^{**} , using that J is an isomorphism.

Therefore, X^{**} is separable - but $X^{**} = (X^*)^*$, so by an earlier result, X^* is separable.

Consequence: $\ell^1(\mathbb{N})$ is not reflexive.

④

Example $X = \mathcal{H}$ a Hilbert space. Claim: $J: \mathcal{H} \rightarrow \mathcal{H}^{**}$ is surjective.

By Riesz lemma, if $\varphi \in \mathcal{H}^*$, then $\exists ! y \in \mathcal{H}$ s.t. $\varphi(x) = \langle x, y \rangle, \forall x \in \mathcal{H}$.
Denote $\varphi_y = y^* \in \mathcal{H}^*$.

So $\Phi: \mathcal{H} \rightarrow \mathcal{H}^*$, $\Phi(y) = y^*$ is a conjugate linear bijection.

Indeed, $\Phi(\alpha_1 y_1 + \alpha_2 y_2) = \varphi_{\alpha_1 y_1 + \alpha_2 y_2} = \overline{\alpha_1} \varphi_{y_1} + \overline{\alpha_2} \varphi_{y_2}$ and Φ is bijective by the Riesz lemma.

Note that $\mathcal{H}^*$ becomes a Hilbert space with inner product

$$\langle w^*, y^* \rangle_{\mathcal{H}^*} := \langle y, w \rangle_{\mathcal{H}} \text{ for } y, w \in \mathcal{H}, y^*, w^* \in \mathcal{H}^*.$$

(Routine verification).

Let $\Psi \in \mathcal{H}^{**}$. By the Riesz lemma applied to $\mathcal{H}^*$, there is a unique $y^* \in \mathcal{H}^*$ s.t. $\Psi = (y^*)^* \neq y^{**}$, meaning that

$$\Psi(z^*) = \langle z^*, y^* \rangle_{\mathcal{H}^*} \text{ for each } z^* \in \mathcal{H}^* = \Phi(\mathcal{H}).$$

We have $y^* = \Phi(y)$ for a unique $y \in \mathcal{H}$.

Claim: $J(y) = \Psi$. (This means exactly that J is surjective)

$$\Leftrightarrow J(y)(z^*) = \Psi(z^*), \quad \forall z^* \in \mathcal{H}^* = \Phi(\mathcal{H})$$

$$\Leftrightarrow z^*(y) = \Psi(z^*), \quad \forall z^* \in \mathcal{H}^*, \text{ by def. of } J$$

$$\Leftrightarrow \langle y, z \rangle_{\mathcal{H}} = \langle z^*, y^* \rangle_{\mathcal{H}^*} \text{ — a —, true}$$

true by def. of inner product on $\mathcal{H}^*$.

Other important examples:

- $L^p(\mathbb{N})$ for $1 < p < \infty$
- $L^p(\Omega, \Sigma, \mu)$ $1 < p < \infty$

Non-examples:

$$l^1(\mathbb{N}), l^\infty(\mathbb{N}) = (l^1(\mathbb{N}))^*.$$

Fact: $l^\infty(\mathbb{N})^*$ is not separable.

Proposition If a Banach space is separable and reflexive, then X^* is separable

Proposition

Suppose that X is a reflexive Banach space. Show that X^* is also reflexive.

Solution Have $J_X: X \rightarrow X^{**}$, $J_X(x)(\varphi) = \varphi(x)$, $x \in X$, $\varphi \in X^*$ is surjective (and always linear, and isometric).

Consider $J_{X^*}: X^* \rightarrow (X^*)^{**}$, $J_{X^*}(\varphi)(\psi) = \psi(\varphi)$, $\varphi \in X^*$, $\psi \in (X^*)^*$.

Now, we can write $\psi = J_X(x)$ for some $x \in X$, with the property that $\psi(\varphi) = J_X(x)(\varphi)$, $\forall \varphi \in X^*$, so $J_{X^*}(\varphi)(J_X(x)) \stackrel{(1)}{=} \varphi(x)$, $\forall \varphi$.

Need to show that J_{X^*} is surjective. Let $\Phi \in (X^*)^{**}$. We must define $\varphi: X \rightarrow \mathbb{C}$ so that $\varphi \in X^*$ and

$$(2) \quad J_{X^*}(\varphi) = \Phi.$$

Now, (2) $\Leftrightarrow J_{X^*}(\varphi)(\psi) = \Phi(\psi)$, $\forall \psi \in (X^*)^* = X^{***}$

$$\xLeftrightarrow{\text{since } J_X \text{ surj.}} J_{X^*}(\varphi)(J_X(x)) = \Phi(J_X(x)), \quad \forall x \in X$$

$$\text{by (1)} \quad \Leftrightarrow \varphi(x) = \Phi(J_X(x)), \quad \forall x \in X.$$

Clearly this defines a linear functional, which is bounded since $|\varphi(x)| \leq \|\Phi\| \cdot \|J_X(x)\| = \|\Phi\| \cdot \|x\|$, $\forall x$, showing $\|\varphi\| \leq \|\Phi\|$.

So, φ can be defined in X^* with $J_{X^*}(\varphi) = \Phi$, showing the desired surjectivity.