

Reflexive spaces

Recall our definition: For a normed space, we defined the map

$$J: X \rightarrow X^{**} \text{ by } J(x): X^* \rightarrow \mathbb{C}, J(x)(f) = f(x) \text{ for every } f \in X^*.$$

We used the notation $J(X) = X^{**}$.

Facts: (i) J is linear and isometric, thus $\|J(x)\| = \|x\|$, $\forall x \in X$

(ii) If X is a Banach space, then $J(X)$ is a closed subspace of X^{**} .

Def: A Banach space X is said to be reflexive if J is surjective (thus, if we have $J(X) = X^{**}$).

Warning: do not confuse it with the existence of some isometric isomorphism between X and X^{**} .

Examples.

(1) Any finite dimensional space. (Since $\dim X^* = \dim X$).

(If $n = \dim X$, suppose $\{e_1, \dots, e_n\}$ is a basis for X , and note that

$\Phi: X^* \rightarrow \mathbb{K}^n$, $\Phi(\varphi) = (\varphi(e_1), \dots, \varphi(e_n))$ for $\varphi \in X^*$ is linear and satisfies $\ker \Phi = \{0\}$. This implies $\dim(X^*) \leq \dim(\mathbb{K}^n) = n = \dim X$).

For each $j = 1, \dots, n$, choose $\varphi_j \in X^*$ s.t. $\varphi_j(e_j) = 1$ and $\varphi_j|_{\text{span}\{e_k, k \neq j\}} = 0$. Then $\{\varphi_1, \dots, \varphi_n\}$ are lin. indep.)

(2) Any Hilbert space $\mathcal{H}$.

Why? We have the Riesz map $\Phi: \mathcal{H} \rightarrow \mathcal{H}^*$, $\Phi(y) := y^*$ with $y^*(x) = \langle x, y \rangle$ for each $x \in \mathcal{H}$. Then Φ is a conjugate linear bijection. Call it $\Phi_{\mathcal{H}}$.

We have an inner product $\langle w^*, y^* \rangle_{\mathcal{H}^*} := \langle y, w \rangle_{\mathcal{H}}$ on $\mathcal{H}^*$. Thus, there is a Riesz map $\Phi^*: \mathcal{H}^* \rightarrow \mathcal{H}^{**}$. Given $\Psi \in \mathcal{H}^{**}$, find the unique y^* s.t. $\Phi^*(y^*) = \Psi$, with $y \in \mathcal{H}$. Verify that $\Psi = J(y)$ (evaluating both sides on arbitrary $z^* \in \mathcal{H}^*$).

(3) Any $L^p(\Omega)$, for (Ω, Σ, μ) measure space, where $1 < p < \infty$.

$$\Psi_p: L^2(\Omega) \rightarrow (L^p(\Omega))^*, \quad \Psi_p(g)(f) = \int_{\Omega} fg d\mu, \quad f \in L^p, g \in L^2$$

$$\Psi_2: L^p(\Omega) \rightarrow (L^2(\Omega))^*, \quad \Psi_2(f)(g) = \int_{\Omega} fg d\mu, \quad \frac{1}{p} + \frac{1}{2} = 1.$$

Reflexivity comes down to the fact that.

$$\forall g \in L^2, \forall f \in L^p: \Psi_p(g)(f) = \Psi_2(f)(g)$$

(4) [Special case of (3)]: $L^p(\mathbb{N})$, $1 < p < \infty$. Here the isometric isomorphism has the form

$$\Psi_p: L^2(\mathbb{N}) \rightarrow (L^p(\mathbb{N}))^*, \quad \Psi_p(y)(x) = \sum_{j=1}^{\infty} x_j y_j \text{ for } x = \{x_j\}_j \in L^p(\mathbb{N}), y = \{y_j\}_j \in L^2(\mathbb{N}).$$

Note: To show surjectivity of Ψ_p in case (3) we need the Radon-Nikodym theorem. For sequence spaces, it is easier due to: (something we already alluded to):

Lemma: Let $X = L^p(\mathbb{N})$ for $1 \leq p < \infty$. Then $\{e^{(j)}\}_{j=1}^{\infty}$ given by $e^{(j)}(k) = \begin{cases} 1 & \text{if } k=j \\ 0 & \text{if } k \neq j \end{cases}, k \in \mathbb{N}$, is a Schauder basis for X .

Proof: Let $x \in L^p(\mathbb{N})$, thus $x = \{x_j\}_{j=1}^{\infty}$, $x_j \in \mathbb{K}$ for all $j \geq 1$ and $\sum_{j=1}^{\infty} |x_j|^p < \infty$. We need to find $\{\alpha_j\}_{j=1}^{\infty} \in \mathbb{K}$ s.t. the series $\sum_{j=1}^{\infty} \alpha_j e^{(j)}$ converges in X with limit x .

Consider the partial sums $\Delta_n = \sum_{j=1}^n \alpha_j e^{(j)} \in L^p$, for $n \geq 1$. Thus

$\Delta_n = (\alpha_1, \alpha_2, \dots, \alpha_n, 0, 0, \dots)$ all 0's. We want $\{\Delta_n\}$ to converge

to $x = (x_1, x_2, \dots, x_n, x_{n+1}, \dots)$ in $\|\cdot\|_p$. We try to set $\alpha_j = x_j$ $\forall j \geq 1$.

Then $\|s_n - x\|_p^p = \sum_{k=1}^{\infty} |s_n(k) - x_k|^p = \sum_{k=n+1}^{\infty} |x_k|^p$. Now, since we

know that $\sum_{j=1}^{\infty} |x_j|^p$ converges (in $\mathbb{K}$), we can for every $\varepsilon > 0$

find $N \in \mathbb{N}$ s.t. $[n \geq N \Rightarrow \sum_{k=n+1}^{\infty} |x_k|^p < \varepsilon]$, that is, $\|s_n - x\|_p^p < \varepsilon$

Whenever $n \geq N$. This is what we wanted. $\square$

Lemma: $\{e^{(j)}\}_{j \geq 1}$ is not a Schauder basis in $l^\infty(\mathbb{N})$.

Proof I: Let $x = \{x_j\}_{j \geq 1} \in l^\infty$ where $x_j = 1$ for all $j \in \mathbb{N}$.

Suppose we could find $\{\alpha_j\} \subseteq \mathbb{K}$ s.t. $\|x - \sum_{j=1}^n \alpha_j e^{(j)}\|_\infty \rightarrow 0$ for $n \rightarrow \infty$.

For each n , $\|x - \sum_{j=1}^n \alpha_j e^{(j)}\|_\infty = \sup_{k \geq 1} \left| x_k - \sum_{j=1}^n \alpha_j e^{(j)}(k) \right|$

$$= \sup \{ |1 - \alpha_1|, \dots, |1 - \alpha_n|, 1, 1, \dots \} \geq 1$$

which cannot go to 0 irrespective of $\{\alpha_j\} \subseteq \mathbb{K}$.

Proof II: Suppose $\{e^{(j)}\}_{j \geq 1}$ is a Schauder basis. Then the set

$\{ \sum_{j=1}^{\infty} \alpha_j e^{(j)} \mid \alpha_j \in \mathbb{Q} \text{ for all } j \geq 1 \}$ is countable and dense in

$\{ \sum \alpha_j e^{(j)} \}$, thus dense in $l^\infty(\mathbb{N})$. In particular, $l^\infty(\mathbb{N})$ will have to be separable.

However, $l^\infty(\mathbb{N})$ is not separable: consider the collection see later

$B = \{x \in l^\infty(\mathbb{N}) \mid x_j \in \{0, 1\}\}$. This is uncountable, and for

any $x, x' \in B$ with $x \neq x'$ we have $\|x - x'\|_\infty = 1$ ($\exists$ some $j \in \mathbb{N}$ so that $x_j \neq x'_j$ because either $x_j = 1$ and $x'_j = 0$ or $x_j = 0$ and $x'_j = 1$).

The collection of open balls of radius $r < \frac{1}{2}$ (e.g. $r = \frac{1}{3}$) centered at elements in B is uncountable and consists of pairwise disjoint

sets: $y \in B_r(x) \cap B_r(x') \Rightarrow 1 = \|x - x'\|_\infty \leq \|x - y\|_\infty + \|y - x'\|_\infty < 2r < 1$, absurd.

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Claim: If A is a dense subset of $\ell^\infty(\mathbb{N})$, then A is uncountable.

Proof: Assume there is $A \subseteq \ell^\infty(\mathbb{N})$ dense. For $x \in B$ we then have

$$B_{\frac{1}{3}}^{\text{open}}(x) \cap A \neq \emptyset,$$

so $\exists y \in A$ s.t. $\|y - x\| < \frac{1}{3}$. Given $x' \neq x$ in B we similarly find $y' \in A$ s.t. $\|y' - x'\| < \frac{1}{3}$. Then

$$1 = \|x - x'\| \leq \|x - y\| + \|y - y'\| + \|y' - x'\| < \frac{2}{3} + \|y - y'\|, \text{ so}$$

$\|y - y'\| > \frac{1}{3}$ and so $y \neq y'$. This establishes an injective "map"

$B \rightarrow A$, thus A must be uncountable.

Why is B uncountable? Essentially, it is in bijection with $\{0,1\}^{\mathbb{N}}$.

Assume not, so assume there is a bijection $\Phi: \mathbb{N} \rightarrow B$ (see exercise 3.15 in [R]). Now we can define a new element in $\ell^\infty(\mathbb{N})$ with entries 0 or 1 that is not equal to any from $\Phi(\mathbb{N})$, a contradiction.

So, we know that for every $j \geq 1$, $\Phi(j)$ is a sequence of 0's and 1's.

Define $\{x_j\}_{j \geq 1}$ by $x_j = \begin{cases} 1 & \text{if } \Phi(j)_j = 0 \\ 0 & \text{if } \Phi(j)_j = 1 \end{cases}$. Then $\{x_j\} \in B$ and

$\{x_j\} \notin \Phi(\mathbb{N})$, contradiction.

Are there examples of non-reflexive Banach spaces? Yes.

(a) $\ell^1(\mathbb{N})$ with $\|\cdot\|_1$. (Known to be non-reflexive.)

(b) $\ell^\infty(\mathbb{N})$ (Known to be non-separable). with $\|\cdot\|_\infty$.

(c) $c_0(\mathbb{N}) = \{ \{x_n\}_n \in \ell^\infty \mid \lim_{n \rightarrow \infty} x_n = 0 \}$ with $\|\cdot\|_\infty$. [$c_0(\mathbb{N})$ is separable (exercise).]

We start with (c). For any $\{x_n\}_n$ in $c_0(\mathbb{N})$, we have

First, we claim that $\Psi: \ell^1(\mathbb{N}) \rightarrow (c_0(\mathbb{N}))^*$, $\Psi(x)(y) = \sum_{n=1}^{\infty} x_n y_n$ for

$x = \{x_n\} \in \ell^1(\mathbb{N})$, $y = \{y_n\} \in c_0(\mathbb{N})$ is an isometric isomorphism (see notes).

Recall also the isometric isomorphism $\Psi_1: \ell^\infty(\mathbb{N}) \rightarrow (\ell^1(\mathbb{N}))^*$.

Use lemma 20.7 to get isom. isom. $(\Psi^{-1})^*$, then compose with Ψ_1 .