

Note that $\Psi^{-1}: (c_0(\mathbb{N}))^* \rightarrow l^1(\mathbb{N})$ satisfies $\Psi^{-1}(\varphi) = \{\varphi(e^j)\}_{j \geq 1}$, by the proof of surjectivity of Ψ . Denote $T = \Psi^{-1}$. (See [B, 3.6.13])

By Lemma 20.7, the map $T^* \circ \Psi_1: l^\infty(\mathbb{N}) \rightarrow (c_0(\mathbb{N}))^{**}$ is an isometric isomorphism. (In [B, Example 3.6.13], our Ψ_1 is 'S').

Clearly, $c_0(\mathbb{N}) \subsetneq l^\infty(\mathbb{N})$. Recall $J: c_0(\mathbb{N}) \rightarrow (c_0(\mathbb{N}))^{**}$ isometric. Now, check that for $y = \{y_n\} \in c_0(\mathbb{N})$ and $\varphi \in c_0(\mathbb{N})^*$, we have

$$(T^* \circ \Psi_1)(y)(\varphi) = J(y)(\varphi) = \sum_{j=1}^{\infty} y_j \varphi(e^j) \in \mathbb{K}.$$

Hence $(T^* \circ \Psi_1)(y) = J(y)$, $\forall y \in c_0(\mathbb{N})$, as elements of $(c_0(\mathbb{N}))^{**}$.

Thus $J(c_0(\mathbb{N})) = (T^* \circ \Psi_1)(c_0(\mathbb{N}))$. Is this all of $(c_0(\mathbb{N}))^{**}$?

We have $(T^* \circ \Psi_1)(l^\infty(\mathbb{N}) \setminus c_0(\mathbb{N})) \neq \emptyset$, so $J(c_0(\mathbb{N})) \subsetneq (c_0(\mathbb{N}))^{**}$.

since $(c_0(\mathbb{N}))^{**} = \underbrace{(T^* \circ \Psi_1)(c_0(\mathbb{N}))}_{= J(c_0(\mathbb{N}))} \cup \underbrace{(T^* \circ \Psi_1)(l^\infty(\mathbb{N}) \setminus c_0(\mathbb{N}))}_{\neq \emptyset}$. So J is not surjective. $\square$

Corollary: ① $l^1(\mathbb{N})$ is not reflexive.

② $l^\infty(\mathbb{N})$ is not reflexive.

For ②, can use that if M is a closed subspace of a reflexive space X , then M is reflexive ($M = c_0(\mathbb{N}) \subseteq l^\infty(\mathbb{N})$), see [B, Theorem 3.6.9].

For ① use that: [X^* reflexive $\Rightarrow X$ reflexive] and small Lemma on next page. (If $l^1(\mathbb{N})$ reflexive, then $(c_0(\mathbb{N}))^*$ will be reflexive, because it is isometrically isom. with $l^1(\mathbb{N})$. Then $c_0(\mathbb{N})$ will be reflexive, contradicting what we just showed.)

Alternative for ②: Use that $l^1(\mathbb{N})^*$ is isometrically isomorphic to $l^\infty(\mathbb{N})$, the small Lemma and ①.

Have shown:

Prop: X reflexive $\Rightarrow X^*$ reflexive.

Theorem: X^* reflexive $\Rightarrow X$ reflexive. (Hahn-Banach)

Spse X^* reflexive and X not reflexive, so $J(X) \subsetneq X^{**}$.

Let $\varphi \in X^{**} \setminus J(X)$ — closed subspace.

By Hahn-Banach (consequence) $\exists \bar{\varphi} \in (X^{**})^*$, $\bar{\varphi} \neq 0$, $\|\bar{\varphi}\|=1$
s.t. $\bar{\varphi}(\varphi) \neq 0$ and $\bar{\varphi}(\psi) = 0$, $\forall \psi \in J(X)$. Thus, for every $x \in X$,

$$\bar{\varphi}(x^{**}) = 0.$$

Now use that $J_{X^{**}}: X^* \rightarrow X^{**} = (X^*)^{**}$ to find $f \in X^*$ s.t.

$J_{X^{**}} f = \bar{\varphi}$. Applying to x^{**} , and remembering that $J_{X^{**}} f = J_{X^*}(f)$ (unique),

$$\text{gives: } 0 = \bar{\varphi}(x^{**}) = J_{X^{**}} f(x^{**}) = X^{**}(f),$$

$$= f(x) \text{ since } x^{**} = J_X(x).$$

$$\Rightarrow f = 0 \text{ in } X^* \Rightarrow \bar{\varphi} = 0, \text{ contradiction. } \square$$

(Small) Lemma Suppose that $T: X \rightarrow Y$ is an isometric isomorphism of Banach spaces and X is reflexive. Then Y is reflexive.

Proof: $T^{**}: X^{**} \rightarrow Y^{**}$

Verify that.

$J_Y \circ T = T^{**} \circ J_X$. This will imply surjectivity of J_Y .

$$(J_Y \circ T)(x)(\varphi) = J_Y(Tx)(\varphi) = \varphi(Tx) = (T^* \circ \varphi)(x)$$

$$(T^{**} \circ J_X)(x)(\varphi) = ((T^*)^* \circ J_X(x))(\varphi) \stackrel{\text{def of } (T^*)^*}{=} J_X(x) \circ T^*(\varphi) \stackrel{\text{def of } J_X}{=} (T^* \varphi)(x) \stackrel{\text{equal}}{=} (T^* \circ \varphi)(x)$$

$$(T^*)^*(F) = F \circ T^* \text{ for } F \in X^{**}.$$

(7)

Proposition We have that $\ell^1(\mathbb{N})$ is the dual of $c_0(\mathbb{N})$. More precisely, the following statements hold:

- (1) For each $x = \{x_j\}_{j \geq 1}$ in $\ell^1(\mathbb{N})$, the map $\varphi_x : c_0(\mathbb{N}) \rightarrow \mathbb{C}$ given by

$$\varphi_x(y) = \sum_{j=1}^{\infty} y_j x_j \text{ for each } y = \{y_j\}_{j \geq 1} \in c_0(\mathbb{N})$$

is a bounded linear functional;

- (2) Every element ϕ in $(c_0(\mathbb{N}))^*$ is of the form φ_x for an element $x = \{x_j\}_{j \geq 1}$ in $\ell^1(\mathbb{N})$;
 (3) The assignment $x \mapsto \varphi_x$ is an isometric isomorphism of $\ell^1(\mathbb{N})$ to $(c_0(\mathbb{N}))^* = B(c_0(\mathbb{N}), \mathbb{C})$.

Proof: That ϕ_x is well-defined, meaning that $\sum_{j=1}^{\infty} y_j x_j$ is summable for any $x \in \ell^1(\mathbb{N})$ and any $y \in c_0(\mathbb{N})$ follows from Hölder's inequality applied to the counting measure on $\mathbb{N}$. Then verify $\varphi_x(ay+bz) = a\varphi_x(y) + b\varphi_x(z)$ for scalars a, b and sequences $y = \{y_j\}, z = \{z_j\}$ converging to zero. From $|\varphi_x(y)| \leq \|x\|_1 \cdot \|y\|_{\infty}$ it follows that φ_x is in $(c_0(\mathbb{N}))^*$ for each fixed x , with $\|\varphi_x\| \leq \|x\|_1$.

Next we show surjectivity, so let $\phi \in (c_0(\mathbb{N}))^*$. We must find a sequence $x \in \ell^1(\mathbb{N})$ such that $\varphi_x(y) = \phi(y)$ for every $y \in c_0(\mathbb{N})$. For every $n \geq 1$, let $e^{(n)}$ be the sequence whose only non-zero entry is the n 'th entry equaling 1, for all $n \geq 1$. Clearly, $e^{(n)} \in c_0(\mathbb{N})$ for each $n \geq 1$. Define $x_n = \phi(e^{(n)})$ for each $n \geq 1$. We claim that $x \in \ell^1(\mathbb{N})$. Choose for every j a complex number α_j of modulus 1 such that $\alpha_j x_j = |x_j|$. Note that for each $N \geq 1$, the sequence $(\alpha_1, \dots, \alpha_N, 0, 0, \dots)$ is in $c_0(\mathbb{N})$ and has norm 1. Using linearity and continuity of ϕ we get

$$\begin{aligned} 0 \leq \sum_{j=1}^N |x_j| &= \sum_{j=1}^N \alpha_j x_j = \sum_{j=1}^N \alpha_j \phi(e^{(j)}) \\ &= \phi\left(\sum_{j=1}^N \alpha_j e^{(j)}\right) = \left|\phi\left(\sum_{j=1}^N \alpha_j e^{(j)}\right)\right| \leq \|\phi\|. \end{aligned}$$

Letting $N \rightarrow \infty$ shows that $\|x\|_1 \leq \|\phi\|$, and proves our claim. Now take $y \in c_0(\mathbb{N})$. For $N \geq 1$ the sequence $y^N = (y_1, \dots, y_N, 0, 0, \dots)$ is in $c_0(\mathbb{N})$. Moreover, you can verify that $\lim_{N \rightarrow \infty} \|y - y^N\|_{\infty} = 0$. By linearity and continuity of ϕ we have

$$\begin{aligned} \phi(y) &= \phi(\lim_N y^N) = \lim_N \phi(y^N) = \lim_N \phi\left(\sum_{j=1}^N y_j e^{(j)}\right) \\ &= \lim_N \sum_{j=1}^N y_j \phi(e^{(j)}) = \lim_N \sum_{j=1}^N y_j x_j = \sum_j x_j y_j = \varphi_x(y). \end{aligned}$$

From here, $\varphi_x = \phi$, showing surjectivity, and $\|x\|_1 \leq \|\phi\| = \|\varphi_x\| \leq \|x\|_1$, showing that the assignment is isometric.