

Exam MAT 4410(2014). Problem 1

Let X be a normed space and $Y \subseteq X$ a subspace.

(1a) Show that $\overline{Y} = \bigcap_{f \in X^* | Y \subseteq \ker f} \ker f$.

Solution: Denote $J = \{f \in X^* | Y \subseteq \ker f\}$. Note that $\ker f$ is a closed subspace of X for each $f \in J$, so $\bigcap_{f \in J} \ker f$ is closed. Denote this set by X_0 . Note that for each $y \in Y$

for each $y \in Y$ and every $f \in J$ we have $y \in \ker f$, so $y \in X_0$. Therefore

$Y \subseteq X_0$, so $\overline{Y} \subseteq \overline{X_0} = X_0$. We next show that $X_0 \subseteq \overline{Y}$.

Assume $y \in X \setminus \overline{Y}$. By a consequence of the Hahn-Banach theorem, there exists $f \in X^*$ with $\|f\|=1$, $f(y) \neq 0$ and $f|_Y = 0$.

$$\text{dist}(y, \overline{Y}) > 0.$$

This means that $Y \subseteq \ker f$, so $f \in J$, but $y \notin \ker f$, thus $y \notin X_0$.

We conclude that $X_0 \subseteq \overline{Y}$, so in all $X_0 = \overline{Y}$.

(1b) Let $1 < p < \infty$ and let $X = \ell^p(\mathbb{N})$.

Let $V = \{x = (x_n) \in X \mid x_n \neq 0 \text{ for finitely many } n \text{'s and } \sum_{n \in \mathbb{N}} x_n = 0\}$.

Show that $\overline{V} = \ell^p(\mathbb{N})$.

Solution By (1a), $\overline{V} = \bigcap_{f \in J} \ker f = X_0$. We know that $X^* \cong \ell^q(\mathbb{N})$, and every $f \in X^*$ is of the form $f = \varphi_y$, $y \in \ell^q(\mathbb{N})$, with $\varphi_y(x) = \sum_{n=1}^{\infty} x_n y_n$ for $x \in X$.

It suffices to show that $X \subseteq X_0$. Since X_0 is an intersection of sets, we must show that $X \subseteq \ker \varphi_y$ for every $\varphi_y \in J$.

In other words, let $\varphi_y|_V = 0$. We will show $\varphi_y = 0$. This means that

we need $y = 0$ in ℓ^q . For $x = (1, -1, 0, \dots) \in V$, $\varphi_y(x) = 0 = y_1 - y_2$, so

we need $y_1 = y_2$. For $x = (1, 0, -1, 0, \dots) \in V$, $\varphi_y(x) = 0 = y_1 - y_3$ so $y_1 = y_3$. We get that $\{y_n\}_n$ is a constant sequence in $\ell^q(\mathbb{N})$, so necessarily $y_n = 0$, then. $\square$