

Spectral theory of general compact operators on Banach spaceWe will use:

- ⊗ [Riesz's lemma: If X is a normed space and $Y \subsetneq X$ is closed, then $\exists x \in X, \|x\|=1, \text{dist}(x, Y) = \inf\{\|x-y\| \mid y \in Y\} \geq \frac{1}{2}$.

To understand invertibility of $T - \lambda I$ for $\lambda \in \mathbb{C}$ when $T \in K(X)$, it suffices to consider $T - I$ where $T \in K(X)$ [since $T - \lambda I = \lambda(\frac{1}{\lambda}T - I)$ for $\lambda \neq 0$].

Proposition 1: Suppose X is a Banach space and $T \in K(X)$. Then $\ker(T - I)$ is finite dimensional. ($\dim \ker(T - I) < \infty$)

Proof: Same idea as for Hilbert spaces, using ⊗ instead of orthogonal decomp. and Pythagoras's.

Set $E := \ker(T - I)$. Assume $\dim E = \infty$. There exists $w_1 \in E$ with $\|w_1\|=1$. Note, $K \cdot w_1 = \text{span}\{w_1\} \subsetneq E$ so by ⊗ $\exists w_2 \in E$ with $\|w_2\|=1$ and $\|w_2 - w_1\| \geq \frac{1}{2}$. We construct $\{w_j\}_{j \geq 1}$.

Induction: Assume we have $\{w_1, \dots, w_n\} \subset E$ with $\|w_j\|=1, \forall j \leq n$ and $\|w_j - w_i\| \geq \frac{1}{2}$ for $1 \leq i \neq j \leq n$. By ⊗ we find $w_{n+1} \in E, \|w_{n+1}\|=1$, and $\|w_{n+1} - w_i\| \geq \frac{1}{2}$ for $1 \leq i \leq n$.

In all, there exists a sequence $\{w_j\}_{j \geq 1} \subset E = \ker(T - I)$ s.t.

$\|w_j\|=1, \forall j$ and $\|w_j - w_i\| \stackrel{(1)}{\geq} \frac{1}{2}$ for $i \neq j$. Compactness of T gives a convergent subsequence $\{Tw_{j_k}\}_k$, but $Tw_j = w_j, \forall j \geq 1$, so $\{w_{j_k}\}$ converges, which is impossible in view of (1).

Corollary If $T \in K(X)$ and $\lambda \neq 0$, then $\dim \ker(T - \lambda I) < \infty$.

11/11/2025 ②

Proposition 2 Let X be a Banach space and $T \in \mathcal{K}(X)$. Then the range of $T - I$ is closed.

Proof: Denote $\text{Ran}(T - I) = (T - I)(X)$.

Let $\{y_n\}_{n \geq 1}$ be a sequence in $\text{Ran}(T - I)$ which converges to some $y \in X$.

So $y_n = (T - I)x_n$ for some $x_n \in X, \forall n \geq 1$. Must show: $y = (T - I)x$ for some $x \in X$. Will write $y_n \stackrel{(2)}{=} (T - I)x_n - (T - I)z_n, z_n \in \ker(T - I)$.

First: Note that if $M \subseteq X$ is fin. dim. then $\forall x' \in X \exists z \in M$ s.t.

$\|x' - z\| = \text{dist}(x', M)$. [Choose sequence $\{z_m\} \subseteq M$ s.t. $d + \frac{1}{m} \geq \|x' - z_m\|, \forall m$, and extract a convergent subsequence from the bounded sequence $\{z_m\}$.

Then $z = \lim z_{m_k}$ has the property we need, by continuity of $\|\cdot\|$.]

Using Prop. 1, choose for every $n \geq 1$ some $z_n \in \ker(T - I)$ so that

$\|x_n - z_n\| = \text{dist}(x_n, \ker(T - I))$. Claim $\{x_n - z_n\}_{n \geq 1}$ is bounded.

Assume we have the claim. By compactness of T , there is a convergent subsequence $\{T(x_{n_k} - z_{n_k})\}_k$, let $w \in X$ be the limit,

From (2), $y_{n_k} \stackrel{(3)}{=} T(x_{n_k} - z_{n_k}) - (x_{n_k} - z_{n_k}), \forall k$, so $\lim_k (x_{n_k} - z_{n_k}) = w - y$.

By continuity of T , we have $\lim_{k \rightarrow \infty} T(x_{n_k} - z_{n_k}) = T(w - y)$.

Taking lim in (3) shows: $y = T(w - y) - (w - y) = (T - I)(\underbrace{w - y}_{=x})$.

It remains to prove the claim. Assume the claim does not hold, so after passing to a subsequence, $\lim_j \|x_{n_j} - z_{n_j}\| = \infty$

Write using (2): $y_{n_j} = T(x_{n_j} - z_{n_j}) - (x_{n_j} - z_{n_j})$ and normalise, so

$$(4) \quad \frac{y_{n_j}}{\|x_{n_j} - z_{n_j}\|} = T\left(\frac{x_{n_j} - z_{n_j}}{\|x_{n_j} - z_{n_j}\|}\right) - \frac{x_{n_j} - z_{n_j}}{\|x_{n_j} - z_{n_j}\|}, \forall j.$$

11/11/2025 ③

Set $w_j = \frac{x_{n_j} - z_{n_j}}{\|x_{n_j} - z_{n_j}\|}$, $\forall j \geq 1$, thus $\{w_j\}$ is a bounded sequence.

Since $\{z_{n_j}\}$ is bounded, the left-hand side of (4) converges to 0, so $\lim_{j \rightarrow \infty} T(w_j) - w_j = 0$. Since by compactness we can assume, after passing to a subsequence and relabelling, that $\{w_j\}$ converges, say to $w \in X$, we obtain: (Triangle inequality: $\|w_j - w\| \leq \|w_j - Tw_j\| + \|Tw_j - w\|$)

$$w = \lim w_j = \lim Tw_j = Tw, \text{ so } w \in \ker(T - I).$$

This shows $\text{dist}(w_j, \ker(T - I)) \rightarrow 0$ as $j \rightarrow \infty$.

$$\text{However, } w_j = \frac{x_{n_j} - z_{n_j}}{\|x_{n_j} - z_{n_j}\|} = \frac{1}{\|x_{n_j} - z_{n_j}\|} (x_{n_j} - z_{n_j}) \underset{\ker(T-I)}{\quad}, \text{ so}$$

$$\text{dist}(w_j, \ker(T - I)) = \frac{1}{\|x_{n_j} - z_{n_j}\|} \text{dist}(x_{n_j}, \ker(T - I)) \rightarrow 1 \text{ by}$$

choice of $\{z_n\}$, giving a contradiction.

Theorem If $T \in K(X)$ and $\ker(T - I) = \{0\}$, then $T - I$ is invertible.

Proof: From the hypothesis we have that $T - I$ is injective. To prove the theorem, it suffices by the Bounded Inverse Theorem to prove that $T - I$ is surjective. In other words, $\text{Ran}(T - I) = X$.

$$\text{For } n \geq 1, (T - I)^n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} T^k \stackrel{\text{of form}}{=} (-1)^n (S_n - I),$$

where $S_n \in K(X)$ (by properties of multiplication and addition of compact operators). We get $\text{Ran}(T - I)^n$ is closed, see Prop. 2.

Set $X_n = \text{Ran}(T - I)^n$, $n \geq 1$. So $(T - I)^2 X = (T - I)(T - I)X$ etc which means: $X \supseteq X_1 \supseteq X_2 \supseteq \dots \supseteq X_n \supseteq X_{n+1} \supseteq \dots$. We will show that this sequence of closed subspaces of X is constant.

11/11/2025 ④

First, we show that the sequence stabilizes at some n .

Assume not, so assume $X_{n+1} \subsetneq X_n$ for every n .

By Riesz's lemma we can find $x_n \in X_n$ with $\|x_n\|=1$ and

$$\text{dist}(x_n, X_{n+1}) \geq \frac{1}{2}, \text{ for each } n \geq 1.$$

$$\begin{aligned} \text{Then } \|Tx_{n+1} - Tx_n\| &= \|Tx_{n+1} \pm x_{n+1} \pm x_n - Tx_n\| \\ &= \|x_n + \underbrace{(T-I)x_{n+1} + x_{n+1} - (T-I)x_n}_{\in (T-I)^{n+1}(X) = X_{n+1}}\| \end{aligned}$$

Similarly, for $k > n$, we obtain

$$\|Tx_k - Tx_n\| = \|x_n + \underbrace{\sum_{j=n}^{k-1} x_{j+1}}_{\in X_{n+1}}\| \geq \frac{1}{2}, \text{ so } \{Tx_n\}_n \text{ cannot have}$$

a Cauchy, therefore convergent, subsequence.

Thus there exists $n \in \mathbb{N}$ st. $X_{n+1} = X_n$.

So $(T-I)^{n+1}(X) = (T-I)^n(X)$, hence $(T-I)^{n+2}(X) = (T-I)^{n+1}(X) = X_n$, so by induction, $X_k = X_n$ for all $k \geq n$.

Claim: $X_n = X$.

Let $x \in X$. Then $(T-I)^n x \in X_n = X_{2n}$, so $\exists y \in X$ st.

$$(T-I)^n x = (T-I)^{2n} y; \text{ therefore } x - (T-I)^n y \in \ker(T-I)^n.$$

But $(T-I)^n$ is injective, because $T-I$ is, so $x - (T-I)^n y = 0$,

giving $x \in X_n$. From $X \supseteq X_1 \supseteq \dots \supseteq X_n = X$ we obtain $X_1 = X$. $\square$

Corollary If $T \in B(X)$ is compact and $\lambda \in \sigma(T)$ with $\lambda \neq 0$, then $\lambda \in \sigma_p(T)$.

Proof: Suppose $\lambda \notin \sigma_p(T)$, so $\ker(T - \lambda I) = \{0\}$. By the previous

theorem, $T - \lambda I = \lambda^{-1}(\frac{1}{\lambda}T - I)$ is invertible, so $\lambda \notin \sigma(T)$, a contradiction.

The Fredholm alternative (the case $\mu=1 \in \mathbb{C}$)

Suppose that $\mathcal{H}$ is a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. We say that $I - T$ satisfies the Fredholm alternative if

either the (inhomogeneous) equation, $(I - T)f = g$ has a unique solution f for a given g in $\mathcal{H}$

or the (homogeneous) equation $(I - T)f = 0$ has a nontrivial solution f in $\mathcal{H}$.

Theorem If $T \in \mathcal{K}(\mathcal{H})$, then $I - T$ satisfies the Fredholm alternative.

Strategy: We already know:

① $\ker(I - T)$ is finite dimensional.

② $\text{Ran}(I - T)$ is closed.

There are two cases: a) $\ker(I - T) = \{0\}$. Then $I - T$ is invertible by the previous theorem, so given $g \in \mathcal{H}$ we find unique $f \in \mathcal{H}$ st. $(I - T)f = g$.

b) $\ker(I - T) \neq \{0\}$. Then there exists $f \neq 0$ in $\ker(I - T)$.

Alternative, see [B, section 4.4]. (More general form.)

$F \in \mathcal{B}(\mathcal{H})$ satisfies the Fredholm alternative if one of the two alternatives occurs:

(a) $\ker F = \ker F^* = \{0\}$ and $F(v) = w$, $F^*(v') = w'$ have a unique solution for all $w, w' \in \mathcal{H}$.

or

(b) $1 \leq \dim \ker F^* = \dim \ker F < \infty$, the equation $F(v) = w$ is consistent $\Leftrightarrow w \in \ker(F^*)^\perp$ and $F^*(v') = w'$ is consistent $\Leftrightarrow w' \in \ker(F)^\perp$.

(Apply to $F = T - \mu I$, $T \in \mathcal{K}(\mathcal{H})$, $\mu \neq 0$ in $\mathbb{C}$.)

11/11/2025 ⑥

One can show:

If $T \in \mathcal{K}(\mathcal{H})$, then $\dim \ker(I-T) = \dim \ker(I-T^*)$, and so are both 0 or finite.

For the proof, enough to show $\dim \ker(I-T) = \dim (\text{Ran}(I-T)^\perp)$

Then use that $\ker(I-T^*) = \text{Ran}(I-T)^\perp$ and, similarly, that $\ker(I-T) = \text{Ran}(I-T^*)^\perp$.

In case $\dim \ker(I-T) = 0$, or $\ker(I-T) = \{0\}$, we already have seen (see the assignment) that $\text{Ran}(I-T)^\perp = \{0\}$. Thus

$\text{Ran}(I-T) = \mathcal{H}$ because we know $\text{Ran}(I-T)$ is closed. So $I-T$ is surjective in this case, hence invertible.

The case $\dim \ker(I-T) \neq 0$.

Assume for a contradiction that $\dim \ker(I-T) < \dim (\text{Ran}(I-T)^\perp)$.

Get injective operator $T_0: \ker(I-T) \rightarrow \text{Ran}(I-T)^\perp$, T_0 not surjective. Extend by 0 to $\mathcal{H}$.
 $\begin{matrix} \psi \\ \mathcal{X} \end{matrix} \mapsto \mathcal{X}$

So $\tilde{T}_0: \mathcal{H} \rightarrow \text{Ran}(I-T)^\perp$ satisfies $\text{Ran } \tilde{T}_0 = \text{Ran } T_0$ and $\ker(I-T^*)$, so finite dimensional (Prop. 1).

$\tilde{T}_0 \in \mathcal{F}(\mathcal{H})$. Set $\tilde{T} = T + \tilde{T}_0$, thus $\tilde{T} \in \mathcal{K}(\mathcal{H})$

Claim: $\ker(I-\tilde{T}) = \{0\}$. By previous, $\text{Ran}(I-\tilde{T})^\perp = \{0\}$, so

$\text{Ran}(I-\tilde{T}) = \mathcal{H}$ and this will imply $\text{Ran } \tilde{T}_0 = \mathcal{H} = \text{Ran } T_0$, a contradiction.

$(I-\tilde{T})x = 0 \Rightarrow x = \tilde{T}x = (T + \tilde{T}_0)x = Tx + \tilde{T}_0x \Rightarrow (I-T)x = \tilde{T}_0x$
 $\Rightarrow (I-T)x \in \underbrace{\text{Ran}(I-T)^\perp}_{=\text{Ran}(T_0)} \cap \text{Ran}(I-T) \Rightarrow (I-T)x = 0 = \tilde{T}_0x \Rightarrow x = 0$.

Thus the claim is proved, leading to the contradiction. $\square$