

MAT 4410 / Exam 2024

① $\mathcal{H}$ Hilbert space over $\mathbb{C}$, $P \in B(\mathcal{H})$ s.t. $P^2 = P$. Let $M = P(\mathcal{H})$.

a) Show M is closed.

Suppose $\{y_n\} \subseteq M$ satisfies $\lim_n y_n = y$. Write $y_n = Px_n$, $x_n \in \mathcal{H}$, then

$$P \text{ cont} \Rightarrow Py = \lim_n Py_n = \lim_n P^2 x_n = \lim_n Px_n = \lim_n y_n = y,$$

thus $y \in M$.

b) If P is compact, then P has finite rank.

We show that the unit ball in $P(\mathcal{H})$ is compact.

Let $\{x_n\}_n \subseteq P(\mathcal{H})$, $\|x_n\| \leq 1$; Then $\exists$ convergent subsequence $\{x_{n_k}\}_k$.

Since $P(\mathcal{H})$ is closed, $\exists x \in \mathcal{H}$ s.t. $\lim x_{n_k} = Px$.

But $x_n = Pz_n$ for some $z_n \in \mathcal{H}$, for each $n \geq 1$, and we have

$$Px = \lim x_{n_k} = \lim P^2 z_{n_k} = \lim Pz_{n_k} = \lim x_{n_k}, \text{ with } \|Px\| \leq 1.$$

Thus $\{x_n\}$ has a convergent subsequence, and we conclude $\dim P(\mathcal{H}) < \infty$.

(c) Assume $\exists U \in B(\mathcal{H})$ s.t. $U^*U = P$.

Show that $P^* = P$ and U is isometric on M . Show $M^\perp \subseteq \ker U$.

Have: $P^* = (U^*U)^* = U^{**}U^* = U^*U = P$.

For $y \in M$, $y = Px$, $x \in \mathcal{H}$, we have

$$\|Uy\|^2 = \langle Uy, Uy \rangle = \langle y, U^*Uy \rangle = \langle y, Py \rangle = \langle y, P^2x \rangle = \langle y, y \rangle = \|y\|^2,$$

thus $U|_M$ is isometric.

Let $z \in M^\perp$. Then $Pz \in M$, so

$$0 = \langle Pz, z \rangle = \langle U^*Uz, z \rangle = \langle Uz, Uz \rangle = \|Uz\|^2, \text{ thus } Uz = 0,$$

showing $z \in \ker U$.

(1d) Assume $V \in B(H)$ is isometric on a closed subspace N of H , and that $N^\perp \subset \ker V$. Let $Q = P_N$. Show that $\langle V^*Vx, x \rangle = \langle Qx, x \rangle, \forall x \in H$.

Deduce that $V^*V = Q$.

Have: for $x \in H$, $x = x_N + x_{N^\perp}$ uniquely according to $H = N \oplus N^\perp$.

$$\begin{aligned} \text{Then } \langle V^*Vx, x \rangle &= \langle Vx, Vx \rangle = \|Vx\|^2 = \|V(x_N + x_{N^\perp})\|^2 = \\ &= \|Vx_N\|^2 = \|x_N\|^2 = \|Qx\|^2 = \langle Qx, x \rangle \text{ since } Q^* = Q. \end{aligned}$$

By polarisation identity, get $\langle V^*Vx, y \rangle = \langle Qx, y \rangle, \forall x, y \in H$, so $V^*V = Q$. Alternatively, use result that says: if S is isometric, then $\langle Sx, Sy \rangle = \langle x, y \rangle, \forall x, y \in H$.

$$\begin{aligned} \text{So } \langle V^*Vx, y \rangle &= \langle (V^*V)^*(V^*V)x, y \rangle = \langle V^*Vx, V^*Vy \rangle = \langle x, y \rangle \\ &= \langle Qx, Qy \rangle = \langle Qx, y \rangle, \forall x, y, \text{ hence } V^*V = Q. \end{aligned}$$

(4) Assume that $T \in B(H)$, for H infinite dimensional, and $\{u_n\}_n$ is an orthonormal sequence in H . Show that $\lim_{n \rightarrow \infty} \|Tu_n\| = 0$.

Suppose $\lim_{n \rightarrow \infty} \|Tu_n\| \neq 0$. This implies that there is a subsequence

$\{Tu_{n_k}\}_k$ converging to some $y \neq 0$ (Since not all subsequences

$\{Tu_{n_k}\}_k$ can converge to 0, and we can extract $\{Tu_{n_{k_j}}\}_{j \in \mathbb{N}}$ convergent by compactness). Relabel and assume $\lim_n Tu_n = y \neq 0$. So $y \in \overline{\text{Ran } T}$.

Recall that $\overline{\text{Ran } T} = (\text{Ran } T)^\perp = (\ker T^*)^\perp$, so $y \in (\ker T^*)^\perp$.

We claim that $y \in \ker T^*$. This will imply $y = 0$, a contradiction.

$$\text{Let } z \in H. \text{ Then } \langle T^*y, z \rangle = \langle y, Tz \rangle = \lim_n \langle Tu_n, Tz \rangle = \lim_n \langle u_n, T^*Tz \rangle,$$

and so $\langle T^*y, z \rangle = 0$ by Bessel's inequality: $\left(\sum_n |\langle u_n, T^*Tz \rangle|^2 \right)$ converges.

Thus $T^*y = 0$, or $y \in \ker T^*$, as claimed.

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(2) Mat H410(2024)

 $\mathcal{H} = \ell^2(\mathbb{N}), \{\delta_k\}_k$ standard o.n.b., $S: \mathcal{H} \rightarrow \mathcal{H}, (S\zeta)(n) = \frac{1}{n}\zeta(n+1)$,
 $\forall \zeta \in \mathcal{H}, n \in \mathbb{N}$.
(2a) Check $S(\delta_1) = 0, S(\delta_k) = \frac{1}{k-1} \delta_{k-1} \forall k \geq 2$. Thus, $\ker S = \text{span}\{\delta_1\}$ Show $S^*(\delta_k) = \frac{1}{k} \delta_{k+1}$ for all $k \geq 1$. $\left[\ker S^* = \{0\} \text{ so } \right.$

$$(S\delta_1)(n) = \frac{1}{n} \delta_1(n+1) = 0 \text{ for all } n \geq 1$$

 $\dim \ker S^* = 0 < \dim \ker S = 1$
 $\Rightarrow S$ fails Fredholm.

$$(S\delta_2)(n) = \frac{1}{n} \delta_2(n+1) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n > 2 \end{cases} = (1 \cdot \delta_1)(1)$$

$$(S\delta_k)(n) = \frac{1}{n} \delta_k(n+1) = \begin{cases} \frac{1}{k-1} & \text{if } n=k-1 \\ 0 & \text{else} \end{cases} = \left(\frac{1}{k-1} \delta_{k-1} \right)(n), \forall n.$$

$$\forall k \geq 2: \langle S\delta_k, \delta_l \rangle = \sum_{n=1}^{\infty} (S\delta_k)(n) \overline{\delta_l(n)} = \sum_{n=1}^{\infty} \left(\frac{1}{k-1} \delta_{k-1} \right)(n) \delta_l(n)$$

$$= (S\delta_k)(l)$$

$$= \left(\frac{1}{k-1} \delta_{k-1} \right)(l)$$

$$= \frac{1}{l} \delta_{l+1}(k)$$

$$= \langle \delta_k, \frac{1}{l} \delta_{l+1} \rangle. \text{ By uniqueness of } S^*, \text{ we get } S^* \delta_l = \frac{1}{l} \delta_{l+1} \forall l.$$

(2b) Show that S is Hilbert-Schmidt. Find $\{\sigma_j\} \in \mathbb{R}_+$ ando.n. sequences $\{u_j\}, \{v_j\}$ s.t. $S\zeta = \sum_{j=1}^{\infty} \sigma_j \langle \zeta, v_j \rangle u_j, \forall \zeta \in \mathcal{H}$.

$$\sum_{j=1}^{\infty} \|S\delta_j\|^2 = \sum_{j=1}^{\infty} \langle S\delta_j, S\delta_j \rangle = \sum_{j=2}^{\infty} \langle \frac{1}{j-1} \delta_{j-1}, \frac{1}{j-1} \delta_{j-1} \rangle =$$

$$= \sum_{j=2}^{\infty} \frac{1}{(j-1)^2} \langle \delta_{j-1}, \delta_{j-1} \rangle = \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty, \text{ so } S \in \text{HS}(\ell^2(\mathbb{N}))$$

$$S^* S \delta_k \stackrel{k \geq 2}{=} S^* \left(\frac{1}{k-1} \delta_{k-1} \right) = \frac{1}{k-1} S^* (\delta_{k-1}) = \frac{1}{k-1} \cdot \frac{1}{k-1} \delta_k = \frac{1}{(k-1)^2} \delta_k, \text{ so}$$

 $\{\delta_k\}$ is an eigenvector for $S^* S$ corresp. to eigenvalue $\frac{1}{(k-1)^2} > 0$. We get.

$$S\zeta = \sum_{j=1}^{\infty} \sqrt{\sigma_j} \langle \zeta, v_j \rangle v_j = \sum_{j=1}^{\infty} \frac{1}{j-1} \langle \zeta, \delta_j \rangle \delta_{j-1}.$$

(4)

For K compact, K^*K is compact s.a., so

$\exists \{\alpha_j\} \subseteq \mathbb{R}_{\geq 0}$ (since $K^*K \geq 0$) and $\{u_j\}$ orthonormal s.t.

$$K^*Kx = \sum_{j=1}^{\infty} \alpha_j \langle x, u_j \rangle u_j. \quad [\text{With } K^*Ku_j = \alpha_j u_j, \forall j.]$$

$\forall x \in \ell^2(\mathbb{N})$

Define: $\sigma_j = \sqrt{\alpha_j} = \frac{1}{\sqrt{j-1}^2} = \frac{1}{j-1}$ for $j \geq 2$, the singular values of K .

Let $v_j = \sigma_j^{-1} K u_j$, j . Since $u_j = \delta_j$ with $\alpha_j = \frac{1}{(j-1)^2}$, we have

$$\text{So } v_j = (j-1)^{-1} S \delta_j = (j-1)^{-1} \cdot \frac{1}{j-1} \delta_{j-1} = \delta_{j-1}. \text{ In all,}$$

for $j \geq 2$

From $\xi = \sum_j \langle \xi, u_j \rangle u_j + \xi_0$ get by continuity of S :

$$\begin{aligned} S\xi &= \sum_{j=2}^{\infty} \langle \xi, u_j \rangle S u_j = \sum_{j=2}^{\infty} \langle \xi, \overset{\delta_j}{u_j} \rangle \frac{1}{j-1} \delta_{j-1} \\ &= \sum_{j=2}^{\infty} \frac{1}{j-1} \langle \xi, \delta_j \rangle \delta_{j-1} = \sum_{j=1}^{\infty} \frac{1}{j} \langle \xi, \delta_{j+1} \rangle \delta_j. \end{aligned}$$

③ (a) $\{y_n\} \subseteq Y$ normed st. $\{\varphi(y_n)\}_n$ bd in $K \forall \varphi \in Y^*$. Show $\{y_n\}$ bd.

Let $T_n: Y^* \rightarrow K$, $T_n(\varphi) = \varphi(y_n)$. Then T_n is linear and bd, and $\sup_n \{T_n(\varphi) | n \geq 1\}$ is bounded in K . $\|y_n\|=1$

PUB $\Rightarrow \sup_n \|T_n\| < \infty$. But $\|y_n\| = \|\varphi_n(y_n)\| = \|T_n(\varphi)\| \leq \|T_n\|$
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so $\sup \|y_n\| \leq \sup \|T_n\| < \infty$.

(b) $T: X \rightarrow Y$ linear st. $\varphi \circ T \in X^*$ for every $\varphi \in Y^*$.

Show: $T \in B(X, Y)$.

Assume not, then $\exists \{x_n\}$ with $\|x_n\| \leq 1$ st. $\|Tx_n\| > n$.

$\Rightarrow T: X \xrightarrow{\varphi \circ T} Y \rightarrow K$

Let $y_n = Tx_n \in Y$, $\forall n$.

Since $\varphi \circ T \in X^*$, get $\{\varphi(Tx_n)\}$ bd in K , $\forall \varphi \in Y^*$, so $\{Tx_n\}$ bd in Y by (a), which contradicts the choice of x_n . $\square$

(c) $T: X \rightarrow Y$ linear, $\|Tx_n\| \leq n$. Show

$\lim \|Tx_n\| = 0$.

$\|Tx_n\| \leq n$