

## MAT 4410 Exam 2014, Problem 4

- a) Given  $(X, \mathcal{B}, \mu)$  measure space with  $\mu(X) < \infty$ , let  $V \subseteq L^\infty(X, \mu)$  be a subspace s.t.  $\|f\|_\infty \stackrel{(1)}{\leq} C\|f\|_2$  for every  $f \in V$  and some  $C > 0$ . Show that  $\dim V \leq C^2 \mu(X)$ .

Solution Note that  $V \subseteq L^2(X, \mu)$  since for  $f \in V$ ,

$$\|f\|_2^2 = \int_X |f(x)|^2 d\mu(x) \leq \int_X \|f\|_\infty^2 d\mu(x) \leq \mu(X) \cdot \|f\|_\infty^2 < \infty.$$

Assume that  $\{f_j\}_{j=1}^n$  is an orthonormal system for  $V$  with  $\|\cdot\|_2$ , where  $n \in \mathbb{N} \cup \{\infty\}$ .

We have: 
$$\sum_{j=1}^n \|f_j\|_2^2 = \sum_{j=1}^n \int_X |f_j(x)|^2 d\mu(x)$$

$$\stackrel{\textcircled{1}}{=} \int_X \sum_{j=1}^n |f_j(x)|^2 d\mu(x) \quad (\text{Monotone convergence})$$

Define  $\varphi_x: V \rightarrow \mathbb{C}$ ,  $\varphi_x(f) = f(x)$  for all  $f \in V$ . Then (1) shows that  $\varphi_x \in V^*$ , where  $V$  has  $\|\cdot\|_2$ , and  $\|\varphi_x\| \leq C$  for every  $x \in X$ .

By the Riesz Lemma,  $\exists g_x \in \overline{V}$ , the closure of  $V$  in  $L^2(X, \mu)$ , s.t.

$$\varphi_x(f) = \langle f, g_x \rangle \text{ for every } f \in V$$

We continue in  $\textcircled{1}$  
$$= \int_X \sum_{j=1}^n |\varphi_x(f_j)|^2 d\mu(x) = \int_X \sum_{j=1}^n |\langle f_j, g_x \rangle|^2 d\mu(x)$$

$$\leq \int_X \|g_x\|_2^2 d\mu(x) \text{ by Bessel's inequality}$$

$$\leq \int_X C^2 d\mu(x) = C^2 \mu(X).$$

So  $n = \sum_{j=1}^n \|f_j\|_2^2 \leq C^2 \mu(X) < \infty$ , showing that  $V$  is finite dimensional.

(2)

b) Assume  $V \subset L^2(X, \mu)$  is inf dim closed subspace. Show that  $V \not\subset L^\infty(X, \mu)$ .

Solution: Assume for a contradiction that  $V \subset L^\infty(X, \mu)$ .

We will try to find  $C$  as in part (a), and so conclude that  $\dim(V) < \infty$ , a contradiction with the hypothesis.

Note: Existence of  $C > 0$  amounts to having the map.

$I_V : (V, \|\cdot\|_2) \rightarrow (V, \|\cdot\|_\infty)$  being bounded.

" $X_1$

" $X_2$

2-Banach spaces.

Need  $V$  to be complete in the  $\|\cdot\|_\infty$  norm.

By the argument on previous page, any Cauchy sequence  $\{f_k\}$  in  $V$  for the  $\|\cdot\|_\infty$  norm is  $L^2$ -convergent to some  $f \in L^2(X, \mu)$ .

and by hypothesis,  $f \in V$ . But a general result says that

we can extract a convergent subsequence  $\{f_{k_\ell}\}_\ell$  for pointwise convergence  $\mu$ -a.e., i.e.

$$f(x) = \lim_{\ell \rightarrow \infty} f_{k_\ell}(x) \quad \mu\text{-a.e.}$$

Then  $\lim_{\ell} \|f_{k_\ell} - f\|_\infty = 0$  with  $f \in V$ , showing that  $(V, \|\cdot\|_\infty)$  is a Banach space.

By the closed graph theorem,  $I_V$  is bounded, and so there exists  $C > 0$  as in part (a).