

Exam 2018

4.  $H$ : infinite-dim. Hilbert space (over  $\mathbb{R}$  or  $\mathbb{C}$ ).

$\{u_n\}_{n \in \mathbb{N}} \subseteq H$  orthonormal basis.

a.)  $\forall T \in B(H)$  and  $k \in \mathbb{N}$ ,

$$\langle T(u_k), T(u_k) \rangle = \sum_{n \in \mathbb{N}} |\langle T(u_k), u_n \rangle|^2.$$

Proof

For any  $x \in H$  we may write

$$x = \sum_{n \in \mathbb{N}} \langle u_n, x \rangle u_n.$$

$$\Rightarrow \langle T(u_k), T(u_k) \rangle = \left\langle \sum_n \langle u_n, T(u_k) \rangle u_n, \sum_m \langle u_m, T(u_k) \rangle u_m \right\rangle$$

linearity  
and cont. of  
 $\langle -, - \rangle$

$$\Downarrow = \sum_n \sum_m \langle u_n, T(u_k) \rangle \overline{\langle u_m, T(u_k) \rangle} \langle u_n, u_m \rangle$$

$$= \sum_n \langle u_n, T(u_k) \rangle \overline{\langle u_n, T(u_k) \rangle}$$

orthonormality  
of  $\{u_n\}_n$

$$= \sum_n |\langle u_n, T(u_k) \rangle|^2$$

$$= \sum_n |\langle T(u_k), u_n \rangle|^2.$$



May also use Parseval's identity:

$$\|x\|^2 = \sum_{n \in \mathbb{N}} |\langle u_n, x \rangle|^2.$$

b.) Let  $P \in B(H)$  be a self-adjoint projection.

Assume  $\langle Pu_k, u_k \rangle \in \{0, 1\}$  for some  $k \in \mathbb{N}$ .

Then  $\langle Pu_k, u_n \rangle = 0 \quad \forall n \in \mathbb{N} \text{ s.t. } n \neq k.$

Proof

By 4a.) we have that

$$\begin{aligned}\langle Pu_k, u_k \rangle &= \langle P^2 u_k, u_k \rangle \\ &= \langle P^* P u_k, u_k \rangle \\ &= \langle Pu_k, Pu_k \rangle \\ &= \sum_{n \in \mathbb{N}} |\langle Pu_k, u_n \rangle|^2\end{aligned}$$

Case 1:  $\langle Pu_k, u_k \rangle = 0$

Then  $\sum_n |\langle Pu_k, u_n \rangle|^2 = 0$ , hence  $\langle Pu_k, u_n \rangle = 0$   
 $\forall n \in \mathbb{N}.$

Case 2:  $\langle Pu_k, u_k \rangle = 1$

$$\begin{aligned}\text{Then } 1 &= \sum_n |\langle Pu_k, u_n \rangle|^2 \\ &= |\langle Pu_k, u_k \rangle|^2 + \sum_{n \neq k} |\langle Pu_k, u_n \rangle|^2 \\ &= 1 + \sum_{n \neq k} |\langle Pu_k, u_n \rangle|^2.\end{aligned}$$

Hence,  $\sum_{n \neq k} |\langle Pu_k, u_n \rangle|^2 = 0 \Rightarrow \langle Pu_k, u_n \rangle = 0 \quad \forall n \in \mathbb{N}$   
s.t.  $n \neq k.$



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c.) Let  $P \in B(H)$  be a self-adjoint projection and define  $D_P \in B(H)$  by linearly extending

$$D_P(u_n) = \langle Pu_n, u_n \rangle u_n \quad \text{for } n \in \mathbb{N}.$$

If  $D_P$  is a projection, then  $D_P = P$ .

Proof

For  $k \in \mathbb{N}$  set  $\lambda_k = \langle Pu_k, u_k \rangle$ . Then  $D_P u_k = \lambda_k u_k$ .

Since  $D_P$  is assumed to be a projection we have that

$$\lambda_k u_k = D_P u_k = D_P^2 u_k = \lambda_k (D_P u_k) = \lambda_k^2 u_k.$$

Hence,  $\lambda_k^2 = \lambda_k \quad \forall k \in \mathbb{N}$ .

Hence,  $\langle Pu_k, u_k \rangle = \lambda_k \in \{0, 1\} \quad \forall k \in \mathbb{N}$ .

By 4b.  $\langle Pu_k, u_n \rangle = 0 \quad \forall k, n \in \mathbb{N} \text{ s.t. } k \neq n$ .

Thus,

$$Pu_k = \sum_{n \in \mathbb{N}} \langle Pu_k, u_n \rangle u_n = \langle Pu_k, u_k \rangle u_k = D_P u_k$$

$\forall k \in \mathbb{N}$ .

Since  $P$  and  $D_P$  agree on a basis of  $H$  and they are both cont., they agree on all of  $H$ .



5. Let  $H$  be an infinite dim. Hilbert space (over  $\mathbb{R}$  or  $\mathbb{C}$ ) and let  $T \in B(H)$  be compact.

a.) What does it mean that  $T$  is compact?

-  $T$  is compact iff  $T(B_1) \subseteq H$  is compact, where  $B_1 = \{x \in H : \|x\| \leq 1\}$ .

-  $T$  is compact iff for every bounded sequence  $\{x_n\}_{n \in \mathbb{N}} \subseteq H$ , the sequence  $\{Tx_n\}_{n \in \mathbb{N}}$  has a convergent subsequence.

b.) For any orthonormal sequence  $\{u_n\}_{n \in \mathbb{N}} \subseteq H$ , the sequence  $\{\langle Tu_n, u_n \rangle\}_{n \in \mathbb{N}}$  converges to 0.

Proof

Assume, to get a contradiction, that  $\{\langle Tu_n, u_n \rangle\}_n$  does not converge to 0.

Then  $\exists \varepsilon > 0$  and  $n_1 < n_2 < \dots$  s.t.

$$|\langle Tu_{n_j}, u_{n_j} \rangle| \geq \varepsilon \quad \forall j \in \mathbb{N}.$$

Set  $x_j := u_{n_j}$ . Then  $\{x_j\}_{j \in \mathbb{N}}$  is a bounded seq., hence  $\{Tx_j\}_{j \in \mathbb{N}}$  has a convergent subsequence,  $\{Tx_{j_k}\}_{k \in \mathbb{N}}$ . Set  $y_k := x_{j_k}$ .

Let  $z \in H$  be s.t.  $Ty_k \rightarrow z$  as  $k \rightarrow \infty$ .

By assumption,  $\exists K \in \mathbb{N}$  s.t.

$$\|Ty_k - z\| < \epsilon/2 \quad \forall k \geq K.$$

Hence,

$$|\langle z - Ty_k, y_k \rangle| \leq \|z - Ty_k\| \|y_k\|$$

$$= \|z - Ty_k\| \quad \leftarrow \text{Recall } y_k = u_{n_{j_k}} \text{ and } \|u_n\| = 1 \quad \forall n \in \mathbb{N}.$$
$$< \epsilon/2.$$

Since  $\{y_k\}_k$  is a subseq. of  $\{x_j\}_j$  we have that  $|\langle Ty_k, y_k \rangle| \geq \epsilon \quad \forall k \in \mathbb{N}$ . Then,

$$|\langle z, y_k \rangle| = |\langle Ty_k, y_k \rangle + \langle z - Ty_k, y_k \rangle|$$

$$\geq |\langle Ty_k, y_k \rangle| - |\langle z - Ty_k, y_k \rangle|$$

$$> \epsilon - \epsilon/2$$

$$= \epsilon/2$$

$$\Rightarrow \sum_{k \in \mathbb{N}} |\langle z, y_k \rangle|^2 \geq \sum_{k \in \mathbb{N}} \frac{\epsilon^2}{4} = \infty. \quad \text{But Bessel's}$$

inequality states that

$$\sum_{k \in \mathbb{N}} |\langle z, y_k \rangle|^2 \leq \|z\|^2 < \infty, \quad \text{giving us our}$$

contradiction.

□

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Exam 2019

4.  $H$ : Nonzero Hilbert space (over  $\mathbb{R}$  or  $\mathbb{C}$ ), and  $I: H \rightarrow H$  denotes the identity.

a.) Assume  $M$  is a closed subspace of  $H$  s.t.  $0 \neq M \neq H$ . Let  $\mathcal{B}$  denote an orthonormal basis for  $M$ , and  $\mathcal{B}'$  denote an orthonormal basis for  $M^\perp$ .

Then  $\mathcal{B} \cup \mathcal{B}'$  is an orthonormal basis for  $H$ .

Proof By assumption any  $u \in \mathcal{B} \cup \mathcal{B}'$  is normalized and if  $u, v \in \mathcal{B} \cup \mathcal{B}'$  s.t.  $u \neq v$  we get that

$$\langle u, v \rangle = 0 \quad \text{by the following}$$

Case 1:  $u, v \in \mathcal{B}$ . Then  $\langle u, v \rangle = 0$  since  $\mathcal{B}$  is an ONB for  $M$ .

Case 2:  $u, v \in \mathcal{B}'$ . Similar to case 1.

Case 3:  $u \in \mathcal{B}, v \in \mathcal{B}'$ . Then in particular  $u \in M$  and  $v \in M^\perp$ , hence  $\langle u, v \rangle = 0$ .

It remains to show that  $\text{Span}(\mathcal{B} \cup \mathcal{B}')$  is dense in  $H$ . Let  $x \in H$ ,  $\epsilon > 0$  be given.

Then  $\exists x_M \in M, x_{M^\perp} \in M^\perp$  s.t.  $x = x_M + x_{M^\perp}$ .

Since  $\text{Span } \mathcal{B} \subseteq M$  is dense, and  $\text{Span } \mathcal{B}' \subseteq M^\perp$  is dense,  $\exists y \in \text{Span } \mathcal{B}, z \in \text{Span } \mathcal{B}'$  s.t.

$$\|x_M - y\| < \epsilon/2, \quad \|x_{M^\perp} - z\| < \epsilon/2.$$

Then  $y+z \in \text{Span}(\mathcal{B} \cup \mathcal{B}')$  and

$$\begin{aligned}\|x - (y+z)\| &= \|x_M + x_{M^\perp} - y - z\| \\ &\leq \|x_M - y\| + \|x_{M^\perp} - z\| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon.\end{aligned}$$

□

b.) Assume  $S \in \mathcal{B}(H)$  is self-adjoint and let  $M$  denote the closed subspace  $\ker(I - S)$ .

Then  $S$  is an isometry iff  $S$  is unitary.

In which case  $S = 2P_M - I$ , where  $P_M$  denotes the orthogonal projection onto  $M$ .

Proof

Any unitary is an isometry. So suppose

$S$  is an isometry. Algebraically this means

that  $S^*S = I$ . Since  $S^* = S$  we get

that  $SS^* = S^2 = S^*S = I$ , hence  $S$  is unitary.

To show that  $S = 2P_M - I$  in this case, we will show that  $P_M = \frac{1}{2}(S + I)$ .

First,  $\frac{1}{2}(S + I)$  is a projection:

$$\frac{1}{2}(S + I)^* = \frac{1}{2}(S + I) \quad \text{since } S^* = S \text{ and } I^* = I.$$

$$\left(\frac{1}{2}(S + I)\right)^2 = \frac{1}{4}(S^2 + 2S + I) = \frac{1}{4}(I + 2S + I) = \frac{1}{2}(S + I).$$

Hence,  $\frac{1}{2}(S+I)$  is a projection. Let  $x \in M$ .

Then  $Sx = x$ , so

$$\frac{1}{2}(S+I)x = \frac{1}{2}(x+x) = x.$$

Meaning  $M \subseteq \text{Im}(\frac{1}{2}(S+I))$ .

Notice that  $(S-I)(S+I) = S^2 - I = 0$ ,

hence  $\frac{1}{2}(S+I)x \in \ker(S-I) = M \quad \forall x \in H$ .

This shows that  $\text{Im}(\frac{1}{2}(S+I)) \subseteq M$ ,

hence

$$M = \text{Im}(\frac{1}{2}(S+I)).$$

Thus,  $\frac{1}{2}(S+I) = P_M$ , completing the proof.  $\square$

5. Let  $H$  be a separable complex Hilbert space.

Exam tip: Very few results in the course requires the Hilbert space in question to be complex. This indicates that we need to use the spectral theorem for self-adjoint (or more generally normal) operators.

a.) Let  $T \in B(H)$  be diagonalizable. Then  
 $\exists S \in B(H)$  s.t.  $S^2 = T$ .

Proof

Let  $\{u_n\}_{n \in \mathbb{N}} \subset H$  be an ONB s.t.

$$Tu_n = \mu_n u_n$$

with  $\mu_n \in \mathbb{C}$  for each  $n \in \mathbb{N}$ . For each  
 $n \in \mathbb{N}$  let  $\lambda_n \in \mathbb{C}$  be s.t.

$$\lambda_n^2 = \mu_n.$$

Define  $S: H \rightarrow H$  by linearly extending

$$Su_n = \lambda_n u_n.$$

Note that  $S \in B(H)$  since  $\{\lambda_n\}_n$  is a bounded  
sequence :

$$\sup_n |\lambda_n| = \sup_n |\lambda_n^2|^{1/2} = \sup_n |\mu_n|^{1/2} = \|T\|^{1/2}.$$

Then, for any  $n \in \mathbb{N}$ ,

$$S^2 u_n = \lambda_n^2 u_n = \mu_n u_n = T u_n,$$

hence  $S^2 = T$ , since  $\{u_n\}_n$  is a  
basis for  $H$ .  $\square$

b.) Let  $T \in \mathcal{K}(H)$  be self-adjoint. Then

$\exists S \in \mathcal{K}(H)$  s.t.  $S^2 = T$ .  $S$  can be chosen to be self-adjoint only if  $T$  is positive.

Proof

By the Spectral Theorem,  $T$  is diagonalizable. Thus,  $\exists S \in \mathcal{B}(H)$  s.t.  $S^2 = T$ , by (a.).

Using the same notation as in (a.) we have an ONB  $\{u_n\}_n$  for  $H$  s.t.

$$Tu_n = \mu_n u_n,$$

$$Su_n = \lambda_n u_n, \quad \forall n \in \mathbb{N}.$$

Since  $T$  is compact,  $|\mu_n| \rightarrow 0$  as  $n \rightarrow \infty$ .

In particular,  $|\lambda_n| = |\mu_n|^{1/2} \rightarrow 0$  as  $n \rightarrow \infty$ , hence  $S$  is compact as well.

If  $T$  is not positive, then  $\exists k \in \mathbb{N}$  s.t.  $\mu_k < 0$ . Note that the equation

$$x^2 = \mu_k$$

has no real solutions in this case,

hence  $\langle Su_k, u_k \rangle \notin \mathbb{R}$  for any  $S$  satisfying  $S^2 = T$  in this case. ⑩

