

§ 9.3 [R] → Known from MAT 3400/4400

Bessel's ineq. (Lemma 9.11)

In an i.p. space V , if $\{e_j\}_{j=1}^{\infty}$ is an o.n. set then
 $\forall x \in V, \quad \sum_{j=1}^{\infty} |\langle x, e_j \rangle|^2 \leq \|x\|^2.$

In particular, the series on the LHS (left-hand side) of the inequality converges.

Lemma 9.12 Given $\mathcal{H}$ Hilbert space and $\{e_j\}_{j=1}^{\infty}$ o.n. set,
 for any sequence $\alpha_1, \alpha_2, \dots$, in $\mathbb{K}$ we have:

$$\sum_{j=1}^{\infty} \alpha_j e_j \text{ converges in } \mathcal{H} \iff \sum_{j=1}^{\infty} |\alpha_j|^2 < \infty.$$

Moreover, $\left\| \sum_{j=1}^{\infty} \alpha_j e_j \right\|^2 = \sum_{j=1}^{\infty} |\alpha_j|^2.$ (This says $\{\alpha_j\}_{j=1}^{\infty} \in \ell^2(\mathbb{N})$)
Parseval's identity

Corollary 9.13 Given $\mathcal{H}$ Hilbert space and $\{e_j\}_{j=1}^{\infty}$ o.n. set,
 the series $\sum_j \langle x, e_j \rangle e_j$ converges for all $x \in \mathcal{H}$.

Of course, we want the limit to be x . For this, we need an o.n. basis.

Definition (Often used). "Classical" definition: A basis in $\mathcal{H}$ is a maximal o.n. set.

Suppose $\mathcal{H}$ is a Hilbert space. A non-empty subset B of $\mathcal{H}$ is a basis for $\mathcal{H}$ if B is an o.n. set and $\text{Span}(B)$ is dense in $\mathcal{H}$.

Remarks: Recall: $\text{Span}(B) = \left\{ \sum_{i \in F} \alpha_i e_i \mid F \text{ finite, } e_i \in B \right\}.$

We refer to such B as an orthonormal basis (o.n.b.) for $\mathcal{H}$ (or just basis)

The case of countable B (finite or countably infinite): ②

Theorem ([R, Prop 9.14]) See MAT 3400/4400. ([Rynne-Youngson, ^{Thm} 3.4])

Let $\mathcal{H}$ be a Hilbert space and $B = \{e_j\}_{j=1}^{\infty}$ a countable o.n. set.

TFAE (the following are equivalent):

- (i) B is a basis for $\mathcal{H}$.
- (*) (ii) $\forall x \in \mathcal{H}, \|x\|^2 = \sum_{j=1}^{\infty} |\langle x, e_j \rangle|^2$ ($\overline{\text{Span}(B)} = \mathcal{H}$)
- (iii) If $\langle x, e_j \rangle = 0$ for all $j \geq 1$, then $x = 0$. ($B^{\perp} = \{0\}$)
- (iv) $\forall x \in \mathcal{H}, \|x\|^2 = \sum_{j=1}^{\infty} |\langle x, e_j \rangle|^2$ (Parseval's identity)
- (v) $\overline{\text{Span}(B)} = \mathcal{H}$.

In [R] this is $\text{clin}(B)$ for "closed linear span".

Example: In $\mathcal{H} = \ell^2 = \ell^2(\mathbb{N}) = \{ \{x_j\}_{j=1}^{\infty} \mid x_j \in \mathbb{K}, \forall j, \sum_{j=1}^{\infty} |x_j|^2 < \infty \}$,
 $\{e^{(j)}\}_{j=1}^{\infty}$ is an o.n.b.

Example: In $L^2[-\pi, \pi]$ with usual Lebesgue measure,

the set $\{ \frac{1}{\sqrt{2\pi}} e^{ikx} \mid k \in \mathbb{Z} \}$ is a basis. In particular,

$\forall f \in L^2, f = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$ where $c_k = \langle f, \frac{1}{\sqrt{2\pi}} e^{ikx} \rangle, k \in \mathbb{Z}$.

Assumed known from MAT 3400/4400:

it contains a
 $\downarrow$ dense sequence

Fact An infinite-dimensional Hilbert space is separable if and only if it contains a countable basis. In this case, any other o.n.b. is countable and $\mathcal{H} \cong \ell^2$ (isometric isomorphism).

Now, $l^2 = l^2(\mathbb{N}) = \{f: \mathbb{N} \rightarrow \mathbb{K} \mid \sum_{j=1}^{\infty} |f(j)|^2 < \infty\}$. (3)

For $\Omega \neq \emptyset$, define $l^2(\Omega) = \{f: \Omega \rightarrow \mathbb{K} \mid \sum_{j \in \Omega} |f(j)|^2 < \infty\}$

Here Ω can be uncountable;
 $\Omega = \mathbb{R}$, or $\Omega = \mathcal{P}(\mathbb{N})$,
 (As summation of a series with positive terms.)

Still have a Hilbert space $l^2(\Omega)$. Q1: Is there a basis B ?

Q2: analogue of (*): $x = \sum_{e \in B} \langle x, e \rangle e$? Q3: Analogue of Parseval's?

Define $e_x = 1_{\{x\}}$, the characteristic function of $x \in \Omega$, set

$B = \{e_x \mid x \in \Omega\}$. Then B is o.n. set: for $x \neq y$ in Ω ,

$$\langle e_x, e_y \rangle = \sum_{z \in \Omega} e_x(z) \overline{e_y(z)} = \delta_{x,y}.$$

Claim: $B^\perp = \{0\}$, where $B^\perp = \{f \in l^2(\Omega) \mid \langle f, b \rangle = 0, \forall b \in B\}$

(We will show that $B^\perp = \{0\} \Leftrightarrow \overline{\text{span}(B)} = \mathcal{H}$, for $\mathcal{H}$ any Hilbert space.

Proof: Let $f \in B^\perp$, let $x \in \Omega$. Then

$$0 = \langle f, e_x \rangle = \sum_{z \in \Omega} f(z) e_x(z) = f(x), \text{ so } f(x) = 0, \forall x \in \Omega.$$

□.

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§2.2 Geometry in Hilbert spaces. Orthogonal decompositions. ⁽⁴⁾

Recall from MAT 3400/4400:

More generally, a
↑ non-empty convex
subset (closed)

Theorem ([B, Corollary 2.2.2])

Let H be a Hilbert space and M a closed subspace. For any $x \in H$ there is a unique $x_M \in M$ s.t.

$$\|x - x_M\| = \inf \{\|x - y\| \mid y \in M\}.$$

Say: x_M is the best approximation to x in M .

Some terminology: for X vector space, M_1, M_2 subspaces,

say: X is the (algebraic) direct sum of M_1, M_2 if $\forall x \in X, \exists$ unique $m_i \in M_i, i=1,2$ s.t. $x = m_1 + m_2$.

If X is a normed space, write $X = M_1 \oplus M_2$ if M_1, M_2 closed and X is their algebraic direct sum.

Lemma: $X = M_1 \oplus M_2 \iff M_1 \cap M_2 = \{0\}$ and $X = M_1 + M_2$.

Proof: " $\Rightarrow$ " For $x \in M_1 \cap M_2$, $x = x + 0 = 0 + x \in X = M_1 \oplus M_2$, so $x = 0$ by uniqueness. Clearly $X = M_1 + M_2$.

" $\Leftarrow$ " Let $x \in X$, then $x = m_1 + m_2$ for some $m_1 \in M_1, m_2 \in M_2$.

If also $x = m_1' + m_2', m_1' \in M_1, m_2' \in M_2$, then

$m_1 - m_1' = m_2' - m_2 \in M_1 \cap M_2$ so $m_1 = m_1'$ and $m_2' = m_2$.

Recall: For $\emptyset \neq S \subseteq (V, \langle \cdot, \cdot \rangle)$, let

$$S^\perp = \{y \in V \mid \langle x, y \rangle = 0, \forall x \in S\}.$$

(5)

Theorem Let H be a Hilbert space and M a closed subspace.

(i) If $x \in H$ and $x_M \in M$ is its best approx in M , then $x - x_M \in M^\perp$.

Moreover, $M^\perp$ is closed and $H = M \oplus M^\perp$, with $x = x_M + (x - x_M)$ the unique decomposition of $x \in H$ with respect to M .

(ii) The map $P_M: H \rightarrow H$ defined by $P_M(x) = x_M$ for $x \in H$ is the orthogonal projection of H on M . It satisfies:

(a) P_M bounded with $\|P_M\| = 1$ if $M \neq \{0\}$.

(b) $P_M^2 = P_M$.

(c) $P_M(H) = M = \text{Ran}(P_M)$ and $\ker P_M = M^\perp$.

(iii) Further, $(M^\perp)^\perp = M$ and $P_{M^\perp} = I - P_M$, $I: H \rightarrow H$ identity map.

Proof: (i) Known from MAT 3400. (Uses the key observation: if $z \in H$, then $z \in M^\perp$ if and only if $\|z - y\| \geq \|z\|$ for all $y \in M$)

(Or, let $y \in M$, let $\alpha > 0$; $\|x - x_M\|^2 \leq \|x - (x_M + \alpha y)\|^2 = \|(x - x_M) + \alpha y\|^2$

See [UR, Prop 10.4] $= \|x - x_M\|^2 - 2\text{Re}\langle x - x_M, \alpha y \rangle + \|\alpha y\|^2$, so

$$2\text{Re}\langle x - x_M, \alpha y \rangle \leq \alpha^2 \|y\|^2, \text{ or } 2\text{Re}\langle x - x_M, y \rangle \leq \alpha \|y\|^2.$$

Let $\alpha \rightarrow 0$ to get $\text{Re}\langle x - x_M, y \rangle = 0$, with $y \in M$ arbitrary.

Similarly, using $x_M + i\alpha y$ will give $\text{Im}\langle x - x_M, y \rangle = 0$.

In all, $\langle x - x_M, y \rangle = 0$, showing $x - x_M \in M^\perp$.

That $M^\perp$ is closed is routine, using continuity of the i.p.

Clearly, $MM^\perp = \{0\}$. By previous lemma, we obtain $H = M \oplus M^\perp$, with every $x \in H$ uniquely decomposing as $x = x_M + (x - x_M) \in M \oplus M^\perp$.

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⑥

(ii) For any $x \in H$, $\|P_M(x)\| = \|x_M\| \leq \|x\|$ since

$$\|x\|^2 = \|x_M\|^2 + \underbrace{\|x - x_M\|^2}_{\geq 0} \text{ by Pythagoras. with } \|y\|=1$$

So $P_M: H \rightarrow H$ is bounded with $\|P_M\| \leq 1$. For $y \in M$ we have

$$y = P_M(y), \text{ so } \|P_M(y)\| = \|y\| = 1 \text{ showing } \|P_M\| = 1.$$

$$x \in H, P_M^2(x) = P_M(P_M(x)) = P_M(x_M) = x_M = P_M(x), \text{ showing } P_M^2 = P_M.$$

Clearly $P_M(H) \subseteq M$. For $y \in M$, $y = P_M(y) \in P_M(H)$ so equality.

If $P_M(x) = 0$ then $x_M = 0$ so $x = x - x_M \in M^\perp$, thus $\ker P_M \subseteq M^\perp$.

Let $y \in M^\perp$. Since $P_M(P_M(y) - y) = 0$, we have $P_M(y) - y \in \ker P_M \subseteq M^\perp$.

$\Rightarrow P_M(y) \in M^\perp$ because $M^\perp$ is a subspace. But $P_M(y) \in P_M(H) = M$,

so $P_M(y) \in M \cap M^\perp$ thus $P_M(y) = 0$ so $y \in \ker P_M$ showing $\ker P_M = M^\perp$.

(iii) First, let $y \in M$, show $y \in (M^\perp)^\perp$. Let $z \in M^\perp$, get $\langle y, z \rangle = 0$

Next, let $y \in (M^\perp)^\perp$ and show $y \in M$. By first part applied to $M^\perp$, get $H = M^\perp \oplus (M^\perp)^\perp$.

Set $y^\perp = y - y_M \in M^\perp$. Then $y = y_M + y^\perp \in M \oplus M^\perp$, so

$$y = y_M + y^\perp \in (M^\perp)^\perp \oplus M^\perp.$$

$$\text{Also } y = y + 0 \in (M^\perp)^\perp \oplus M^\perp.$$

$\Rightarrow y = y_M \in M$
by uniqueness of decomposition.

$$\text{So } (M^\perp)^\perp = M.$$

To show $P_{M^\perp} = I - P_M$, let $x \in H$, then $x = (x - x_M) + x_M \in M^\perp \oplus (M^\perp)^\perp$

$$\text{So } P_{M^\perp}(x) = x - x_M = I(x) - P_M(x) = (I - P_M)(x), \text{ as needed. } \square$$