

MAT 4410 Lecture 4.

Recall $\mathcal{H} = M \oplus M^\perp$ whenever M is a closed subspace of a Hilbert space $\mathcal{H}$.

Corollary ①: If $M \subseteq \mathcal{H}$ closed, then $M = \mathcal{H} \Leftrightarrow M^\perp = \{0\} \Leftrightarrow P_M = I$.

Corollary ②: If S is a nonempty subset of $\mathcal{H}$, then $\text{span}(S)$ is dense

Example: $L^2(\Omega, \mathcal{A}, \mu)$, $E \in \mathcal{A}$, $F = E^c$. $\{ \text{in } \mathcal{H} \Leftrightarrow S^\perp = \{0\}$.

Define $M_E := \{g \in L^2 \mid g \text{ lives essentially on } E\}$

Proof: Set $M = \overline{\text{span}(S)}$.

Verify that $M^\perp = S^\perp$ to conclude.

ie. $\mu(\{x \in F \mid g(x) \neq 0\}) = 0$.

Then $M_F = (M_E)^\perp$ and $M_E = (M_F)^\perp$, see [B, Ex. 2.2.9].

Further, $P_{M_E}(f) = f \cdot 1_E$, $\forall E \in \mathcal{A}$; $P_{M_E}^\perp = f \cdot 1_F$.

Exercise 10.9: (1) For $X \subseteq \mathcal{H}$ linear subspace, show that

$$(X^\perp)^\perp = \overline{X}.$$

(2) For $E \subseteq \mathcal{H}$, $E^\perp = [\text{cln}(E)]^\perp$, $E \neq \emptyset$.

Let $\mathcal{H}$ Hilbert space, assume J is a non-empty index set, and suppose $B = \{u_j\}_{j \in J}$ is a collection (set) of vectors $u_j \in \mathcal{H}$ s.t.

$$\langle u_j, u_k \rangle = \delta_{j,k} \text{ for all } j, k \in J.$$

Let $x \in \mathcal{H}$. For each finite subset $K \subseteq J$, we have

$$\sum_{j \in K} |\langle x, u_j \rangle|^2 \leq \|x\|^2 \text{ by Bessel's inequality.}$$

(Choose an enumeration $K = \{k_1, \dots, k_m\}$).

Define $\sum_{j \in J} |\langle x, u_j \rangle|^2 \stackrel{(1)}{=} \sup \left\{ \sum_{j \in K} |\langle x, u_j \rangle|^2 \mid K \subseteq J \text{ finite} \right\}$.

Lemma (Generalised Bessel's inequality)
 $\forall x \in \mathcal{H}, \sum_{j \in J} |\langle x, u_j \rangle|^2 \leq \|x\|^2$. Question: $\sum_{j \in J} |\langle x, u_j \rangle|^2 = \|x\|^2$?

Lemma: Given an orthonormal set $B = \{u_j\}_{j \in J}$, the set $\{j \in J \mid |\langle x, u_j \rangle| \neq 0\}$ is countable for all $x \in \mathcal{H}$.

Proof: Let $x \in \mathcal{H}$. Have:

$$\begin{aligned} \{j \in J \mid |\langle x, u_j \rangle| \neq 0\} &= \{j \in J \mid |\langle x, u_j \rangle| > 0\} \\ &= \bigcup_{n \in \mathbb{N}} \{j \in J \mid |\langle x, u_j \rangle| \geq \frac{1}{n}\}. \end{aligned}$$

Claim: if $\varepsilon > 0$, then $\{j \in J \mid |\langle x, u_j \rangle| \geq \varepsilon\}$ is finite.

Indeed, assume the contrary, then there is a sequence of indices $\{i_k\}_{k \in \mathbb{N}}$ in J s.t. $|\langle x, u_{i_k} \rangle| \geq \varepsilon$ for all $k \in \mathbb{N}$.

This implies that $\sum_{k=1}^{\infty} |\langle x, u_{i_k} \rangle|^2$ is not bounded, which contradicts Bessel's inequality. This shows the claim.

Apply the claim with $\varepsilon = \frac{1}{n}$ for each $n \geq 1$ and use the fact that a countable union of finite sets is countable. $\square$

Consequence: for J arbitrary (possibly uncountable) 27.08.2022
 and $\{u_j\}_{j \in J} \subseteq \mathcal{H}$ orthonormal, we have for every $x \in \mathcal{H}$ that ③

there is $\{j_1, j_2, j_3, \dots\} \subseteq J$ countable such that $\sum |\langle x, u_j \rangle|^2$, defined as a supremum in $(\mathbb{R})$, will equal $\sum_{k=1}^{\infty} |\langle x, u_{j_k} \rangle|^2$.

Lemma For every $x \in \mathcal{H}$, the series $\sum_{j \in J} \langle x, u_j \rangle u_j$ converges and its limit, call it $y \in \mathcal{H}$, satisfies $\langle y, u_j \rangle = \langle x, u_j \rangle$ for all $j \in \{j_1, j_2, \dots\}$.

Proof Let $s_n = \sum_{k=1}^n \langle x, u_{j_k} \rangle u_{j_k}$, then for $n > m$ we have

$$\begin{aligned} \|s_n - s_m\|^2 &= \left\| \sum_{k=m+1}^n \langle x, u_{j_k} \rangle u_{j_k} \right\|^2 \\ &= \sum_{k=m+1}^n |\langle x, u_{j_k} \rangle|^2 \text{ by generalized Pythagoras.} \end{aligned}$$

Since this can be made $< \varepsilon$ for arbitrary $\varepsilon > 0$ when $n > m \geq n_\varepsilon$, some $n_\varepsilon \in \mathbb{N}$, we get

that $\{s_m\}_m$ is a Cauchy seq. in $\mathcal{H}$, thus convergent to $y \in \mathcal{H}$.

Note: The limit will not depend on the order of summation, since $\sum_{j \in J} |\langle x, u_j \rangle|^2$ is absolutely convergent (by Bessel's ineq).

Gram-Schmidt orthonormalisation:

Let $\mathcal{H}$ Hils. space, $\mathcal{H} \neq \{0\}$, and suppose $\{x_j\}_{j \in \mathbb{N}} \subseteq \mathcal{H} \setminus \{0\}$ is such that $\text{Span}\{x_j\} = \mathcal{H}$. Then there is an o.n. sequence $\{u_j\}_{j \in \mathbb{N}}$ with $\text{Span}\{u_j\} = \text{Span}\{x_j\}$.

(4)

Theorem Let $\mathcal{H}$ be a (non-zero) Hilbert space and $\mathcal{B}$ an orthonormal subset of $\mathcal{H}$. TFAE:

- (a) $\mathcal{B}$ is an orthonormal basis for $\mathcal{H}$. (i.e. $\mathcal{B}^\perp = \{0\}$)
 (b) Every $x \in \mathcal{H} \setminus \{0\}$ has a Fourier expansion

$$(1) \quad x = \sum_{u \in B_x} \langle x, u \rangle u,$$

where $B_x = \{u \in \mathcal{B} \mid \langle x, u \rangle \neq 0\}$ is countable and nonempty.

- (c) For every $x \in \mathcal{H}$, we have $\|x\|^2 = \sum_{u \in \mathcal{B}} |\langle x, u \rangle|^2$. (Parseval's inequality)

Remark: (b) means that for every enumeration $\{u_j \mid j \in \mathbb{N}\}$ of the distinct elements in B_x we have (or $u_j, k \in \mathbb{N}$ above)

$$\lim_{n \rightarrow \infty} \|x - \sum_{j=1}^n \langle x, u_j \rangle u_j\| = 0.$$

Proof: (a) $\Rightarrow$ (b). Assume $\mathcal{B}$ is an o.n. basis. Thus $\text{Span}(\mathcal{B}) = \mathcal{H}$. By Corollary 1, $\mathcal{B}^\perp = \{0\}$.

Let $x \in \mathcal{H} \setminus \{0\}$. Then B_x countable by earlier lemma.

If $B_x = \emptyset$ then $\langle x, u \rangle = 0$ for all $u \in \mathcal{B}$ so $x \in \mathcal{B}^\perp = \{0\}$, a contradiction. So we have $B_x \neq \emptyset$. Assume B_x countably infinite, let $\{u_j \mid j \in \mathbb{N}\}$ be an enumeration of its (distinct) elements, thus we have $\sum_{j=1}^{\infty} |\langle x, u_j \rangle|^2 \leq \|x\|^2$.

As explained in earlier lemma, $\left\{ \sum_{j=1}^n \langle x, u_j \rangle u_j \right\}_{n=1}^{\infty}$ converges

Let $y = \sum_{j=1}^{\infty} \langle x, u_j \rangle u_j$ be the limit. Claim $y = x$.

Will show $y - x \in \mathcal{B}^\perp$, which will give the claim.

For $b \in B \setminus B_x$, $\langle x, b \rangle = 0$ and (5)

$$\langle y, b \rangle = \left\langle \sum_{j=1}^{\infty} \langle x, u_j \rangle u_j, b \right\rangle = \sum_{j=1}^{\infty} \langle x, u_j \rangle \underbrace{\langle u_j, b \rangle}_{=0} = 0.$$

so $\langle y - x, b \rangle = 0$. by continuity of i.p.

$$\text{For } u_k \in B_x, \quad \langle y, u_k \rangle = \left\langle \sum_{j=1}^{\infty} \langle x, u_j \rangle u_j, u_k \right\rangle = \sum_{j=1}^{\infty} \langle x, u_j \rangle \langle u_j, u_k \rangle = \langle x, u_k \rangle, \text{ so } \langle y - x, u_k \rangle = 0.$$

Since b was arbitrary, we get $\langle y - x, b \rangle = 0$ for all $b \in B$, as wanted.

(b) $\Rightarrow$ (c). Let $x \in H \setminus \{0\}$, assume B_x countably infinite, write it as $\{u_j\}_{j \in \mathbb{N}}$. By Pythagoras's and continuity of norm,

$$\begin{aligned} \|x\|^2 &= \langle x, x \rangle = \left\langle \sum \langle x, u_j \rangle u_j, \sum \langle x, u_j \rangle u_j \right\rangle \\ &= \sum_{j=1}^{\infty} |\langle x, u_j \rangle|^2 \stackrel{\text{Want}}{=} \sum_{u \in B} |\langle x, u \rangle|^2. \end{aligned}$$

Let $\varepsilon > 0$. There is $n \in \mathbb{N}$ s.t.

$$\|x\|^2 - \sum_{j=1}^n |\langle x, u_j \rangle|^2 < \varepsilon.$$

So $\|x\|^2 - \varepsilon < \sum_{j=1}^n |\langle x, u_j \rangle|^2 \leq \sum_{u \in B} |\langle x, u \rangle|^2$. Now let $\varepsilon \rightarrow 0$.

Together with (generalised) Bessel's inequality, we get

$$\|x\|^2 = \sum_{u \in B} |\langle x, u \rangle|^2 \text{ for } x \neq 0. \text{ For } x=0 \text{ it is obvious.}$$

(c) $\Rightarrow$ (a) Will show $B^\perp = \{0\}$.

Let $x \in B^\perp$. Then $\langle x, u \rangle = 0$ for all $u \in B$, so by Parseval's

identity in (c) we see that $x=0$ (since $\|x\|=0$)

Read Example 2.3.10

