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MAT 4410 Lecture 5

The adjoint on a bounded operator on a Hilbert space

[R, §13.17], [B, Chp. 2.4].

Motivation: Consider the following two examples.Example 1: Let $H = \mathbb{K}^n$ with i.p. $\langle x, y \rangle = \sum_{j=1}^n x_j \bar{y}_j$, $x, y \in \mathbb{K}^n$.Suppose $A = [a_{kj}]_{1 \leq j, k \leq n}$ is a matrix in $M_n(\mathbb{K})$ and let $T = T_A$, i.e. $TX = y$, where for $x = (x_1, x_2, \dots, x_n)^t$ we have $y_k = \sum_{j=1}^n a_{kj} x_j$.Write $A = [a_{kj}]_{1 \leq j, k \leq n}$

$$\text{row } k \rightarrow \begin{bmatrix} a_{k1} & \dots & a_{kj} & \dots & a_{kn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^n a_{kj} x_j \end{bmatrix} \leftarrow y_k.$$

What is the operator with matrix

$$\begin{bmatrix} \overline{a_{11}} & \dots & \overline{a_{k1}} & \dots & \overline{a_{n1}} \\ \overline{a_{12}} & \dots & \overline{a_{k2}} & \dots & \overline{a_{n2}} \\ \vdots & & \vdots & & \vdots \\ \overline{a_{1n}} & \dots & \overline{a_{kn}} & \dots & \overline{a_{nn}} \end{bmatrix} = [\overline{a_{jk}}]_{1 \leq j, k \leq n} = \overline{A}^t ?$$

Example 2Suppose $(\Omega, \mathcal{A}, \mu)$ is a measure space, suppose $f \in L^\infty(\Omega, \mathcal{A}, \mu)$.Let $M_f: H \rightarrow H$, $H = L^2(\Omega, \mathcal{A}, \mu)$, $M_f g = fg$ for $g \in H$.What is the relation between M_f and the operator $g \mapsto \bar{f}g$?

Answer: the adjoint operator.

Recall: if X is a normed space, then its dual space is $X^* := \{ \varphi: X \rightarrow \mathbb{K} \mid \varphi \text{ linear, bounded} \}$. Fact: X^* is a Banach space.

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Recall important result from MAT3400/4400

Theorem (The Riesz lemma)

Let H be a Hilbert space over $\mathbb{C}$. For each $y \in H$, let $\ell_y: H \rightarrow \mathbb{C}$ be given by $\ell_y(x) := \langle x, y \rangle$ for all $y \in H$.
Then $\ell_y \in H^*$.

For every $\ell \in H^*$ there is a unique $y \in H$ s.t. $\ell(x) = \ell_y(x)$ for all $x \in H$. Thus the map $y \mapsto \ell_y$ is a bijection from H onto H^* . Moreover, this map is isometric, meaning $\|y\| = \|\ell_y\|$ for all $y \in H$, and is conjugate linear, meaning

$$(2) \ell_{\alpha_1 y_1 + \alpha_2 y_2} = \overline{\alpha_1} \ell_{y_1} + \overline{\alpha_2} \ell_{y_2}, \quad \forall \alpha_1, \alpha_2 \in \mathbb{C}, y_1, y_2 \in H.$$

(Fact: can replace $\mathbb{C}$ with $\mathbb{K}$).

Theorem Given $T \in B(H)$, there is a unique operator in $B(H)$, denoted T^* , satisfying

$$(3) \langle Tx, y \rangle = \langle x, T^*y \rangle, \quad \forall x, y \in H.$$

Proof: Need: for each $y \in H$, must find a new vector, which will be denoted T^*y , s.t. we can fill in $\langle Tx, y \rangle = \langle x, ? \rangle$ for any $x \in H$. Let's look at the map $\ell: x \mapsto \langle Tx, y \rangle$ from H to $\mathbb{C}$. It is:

linear, $\langle T(\alpha_1 x_1 + \alpha_2 x_2), y \rangle = \langle \alpha_1 Tx_1 + \alpha_2 Tx_2, y \rangle = \alpha_1 \langle Tx_1, y \rangle + \alpha_2 \langle Tx_2, y \rangle$
for all $x_1, x_2 \in H, \alpha_1, \alpha_2 \in \mathbb{C}$.

Bounded: $|\langle Tx, y \rangle| \leq \|Tx\| \cdot \|y\|$ by the Cauchy-Schwarz ineq.
 $\leq \|T\| \cdot \|x\| \cdot \|y\|$ by assumption on T .

Taking supremum over all $x \in \mathcal{H}$ with $\|x\| \leq 1$ we 2/09/2025
(3)

see that $\varphi \in \mathcal{H}^*$ and $\|\varphi\| \leq \|T\| \cdot \|y\|$.

By the Riesz lemma, $\exists$ unique vector $z \in \mathcal{H}$ s.t. $\varphi = \varphi_z$, or $\varphi(x) = \varphi_z(x)$, $\forall x \in \mathcal{H}$.

So $\langle Tx, y \rangle = \langle x, z \rangle$, $\forall x \in \mathcal{H}$. Denote $T^*y = z$. This defines the map $T^*: \mathcal{H} \rightarrow \mathcal{H}$ with property (3).

Show now that T^* is linear. Let $\alpha_1, \alpha_2 \in \mathbb{K}$, $y_1, y_2 \in \mathcal{H}$.

By (3) in Riesz' lemma; $\forall x \in \mathcal{H}$,

$$\begin{aligned} \langle x, T^*(\alpha_1 y_1 + \alpha_2 y_2) \rangle &= \varphi_{\alpha_1 y_1 + \alpha_2 y_2}(x) \stackrel{(2)}{=} \overline{\alpha_1} \varphi_{y_1}(x) + \overline{\alpha_2} \varphi_{y_2}(x) \\ &= \overline{\alpha_1} \langle x, T^* y_1 \rangle + \overline{\alpha_2} \langle x, T^* y_2 \rangle \\ &= \langle x, \overline{\alpha_1} T^* y_1 + \overline{\alpha_2} T^* y_2 \rangle \\ &= \langle x, \alpha_1 T^* y_1 + \alpha_2 T^* y_2 \rangle. \end{aligned}$$

Thus, $T^*(\alpha_1 y_1 + \alpha_2 y_2) - (\alpha_1 T^* y_1 + \alpha_2 T^* y_2) \in \mathcal{H}^\perp$, but $\mathcal{H}^\perp = \{0\}$, so get linearity of T^* .

For boundedness, $\|T^*y\| = \|z\| = \|\varphi_z\| = \|\varphi\| \leq \|T\| \cdot \|y\|$.

Take supremum over y with $\|y\| \leq 1$, thus $\|T^*\| \leq \|T\|$, so $T^* \in B(\mathcal{H})$.

If $\langle Tx, y \rangle = \langle x, T^*y \rangle = \langle x, Sy \rangle$, $\forall y$, for $S \in B(\mathcal{H})$, then $Sy - T^*y \in \mathcal{H}^\perp$ so $Sy = T^*y$, $\forall y$.

Proposition: For $S, T \in B(\mathcal{H})$, and $\alpha, \beta \in \mathbb{K}$, we have □.

(i) $(\alpha S + \beta T)^* = \overline{\alpha} S^* + \overline{\beta} T^*.$

(ii) $(ST)^* = T^* S^*.$

(iii) $(T^*)^* = T.$

(iv) $\|T\| = \|T^*\|.$

(v) $\|T^* T\| = \|T\|^2.$ (the C^* -identity)

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Proof:

(i) Note $\langle (S)x, y \rangle = \langle x, (S)^*y \rangle$, $\forall x \in H$ by def. of $(S)^*$.

$$\begin{aligned} \parallel \\ \alpha \langle Sx, y \rangle &= \alpha \langle x, S^*y \rangle, \quad \forall x \in H \text{ by def of } S^* \\ &= \langle x, \overline{\alpha(S^*y)} \rangle, \quad \forall x \in H. \end{aligned}$$

Conclude that $(\alpha S)^* = \overline{\alpha} S^*$ and finish (i).

(ii) Exercise.

(iii) Idea: use uniqueness of the adjoint. Given T , let $x, y \in H$.

Then $\langle T^*x, y \rangle = \langle x, (T^*)^*y \rangle$ by def. of $(T^*)^*$.

$$\begin{aligned} \parallel \\ \overline{\langle y, T^*x \rangle} &= \overline{\langle Ty, x \rangle} \text{ by def of } T^* \\ &= \langle x, Ty \rangle. \end{aligned}$$

So $(T^*)^*y - Ty \in H^\perp$, $\forall y \in H$,
showing $(T^*)^* = T$.

(iv) Have seen $\|T^*\| \leq \|T\|$. Need converse. Let $x \in H$.

$$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, T^*(Tx) \rangle = |\langle x, T^*(Tx) \rangle|$$

Cauchy-Schwarz $\leq \|x\| \cdot \|T^*(Tx)\| \leq \|x\| \cdot \|T^*\| \cdot \|Tx\|$. For $Tx \neq 0$,
get $\|Tx\| \leq \|T^*\| \cdot \|x\|$, so $\|T\| \leq \|T^*\|$. Thus $\|T\| = \|T^*\|$.

(v) $\|T^*Tx\| \leq \|T^*\| \cdot \|Tx\| \leq \|T^*\| \cdot \|T\| \cdot \|x\| = \|T\|^2 \cdot \|x\|$, so

$$\|T^*T\| \leq \|T\|^2. \text{ Need converse.}$$

$$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, T^*Tx \rangle \leq \|x\| \cdot \|T^*Tx\| \leq \|T^*T\| \cdot \|x\|^2.$$

For $x \neq 0$, get $\|T\|^2 \leq \|T^*T\|$, as wanted. $\square$

We call T^* the adjoint of T .

Example (3) $\mathcal{H}$ sep. with $B = \{u_j\}_{j \in \mathbb{N}}$ o.m. basis. 2/09/2025 ⑤

Let $\{\lambda_j\}_{j \in \mathbb{N}}$ be a bounded seq. in $\mathbb{K}$. Denote $M = \sup_{j \in \mathbb{N}} |\lambda_j| < \infty$.

Define $D: \mathcal{H} \rightarrow \mathcal{H}$, $Du_j = \lambda_j u_j, \forall j$.

Then: $D \in B(\mathcal{H})$ and $D^* u_j = \overline{\lambda_j} u_j, \forall j \in \mathbb{N}$.

Existence: of D : take $x \in \mathcal{H}$. Then $\forall n \in \mathbb{N}$,

$$\sum_{j=1}^n |\lambda_j \langle x, u_j \rangle|^2 \leq \sum_{j=1}^n |\lambda_j|^2 |\langle x, u_j \rangle|^2 \leq M^2 \sum_{j=1}^n |\langle x, u_j \rangle|^2.$$

By Bessel's ineq, $\sum_{j=1}^n |\lambda_j \langle x, u_j \rangle|^2 \leq M^2 \|x\|^2$, so letting $n \rightarrow \infty$ get

$\sum_{j=1}^{\infty} \lambda_j \langle x, u_j \rangle u_j$ converges in $\mathcal{H}$. Let $Dx = \sum_{j=1}^{\infty} \lambda_j \langle x, u_j \rangle u_j$.

Moreover, by earlier lemma, $\langle Dx, u_k \rangle = \lambda_k \langle x, u_k \rangle, \forall k \in \mathbb{N}$.

By Parseval's ineq, $\|Dx\|^2 = \sum_j |\langle Dx, u_j \rangle|^2 = \sum_j |\lambda_j \langle x, u_j \rangle|^2 \leq M^2 \|x\|^2$.

In all, $D \in B(\mathcal{H})$ with $\|D\| \leq M$.

From $\|D\| \geq \|Du_k\| = \|\lambda_k u_k\| = |\lambda_k|, \forall k$ we get $\|D\| \geq M$, so equal.

Call D the diagonal operator associated to $\{\lambda_j\}_{j \in \mathbb{N}}$ w.r.t.

basis B . What is D^* ? Let $F \in B(\mathcal{H})$ be the operator associated to $\{\overline{\lambda_j}\}_{j \in \mathbb{N}}$ as in the situation above. Show: $D^* = F$.

For $k \in \mathbb{N}$, $x \in \mathcal{H}$, we have $\langle x, F(u_k) \rangle = \langle x, \overline{\lambda_k} u_k \rangle = \overline{\lambda_k} \langle x, u_k \rangle$.

and $\langle x, D^*(u_k) \rangle = \langle Dx, u_k \rangle = \langle \sum_j \lambda_j \langle x, u_j \rangle u_j, u_k \rangle = \lambda_k \langle x, u_k \rangle$.

So $F(u_k) = D^*(u_k)$, for $k \in \mathbb{N}$. Since $B = \{u_k\}_k$ is a basis, it follows that $F = D^*$. /Or, show $\langle x, F(y) \rangle = \langle x, D^*(y) \rangle, \forall x, y \in \mathcal{H}$.