

MAT 4410 Lecture 6 [Plan: [B, 2.4.-2.5], [R, 13.1-13.2] B109/2021 ①  
 Have seen ①  $\langle Tx, y \rangle = \sum_{k=1}^n (Tx)_k \overline{y_k} = \sum_{k=1}^n \left( \sum_{j=1}^n a_{kj} x_j \right) \overline{y_k}$   
 [B, Ex. 2.4.4]  $= \sum_{j=1}^n x_j \left( \sum_{k=1}^n \overline{a_{kj}} y_k \right) = \langle x, T^* y \rangle$ , where  $T^* \in B(\mathbb{K}^n)$   
 is given by  $(T^* y)_j = \sum_{k=1}^n \overline{a_{kj}} y_k$ ,  $j=1, \dots, n$ .

② [B, Ex. 2.4.7],  $M_f(g) = fg$ , get  $H = L^2(\Omega, \mathcal{A}, \mu)$ ,  $f \in L^\infty$ . Then  $M_f^* = M_{\overline{f}}$ ,  
 where  $\overline{f}$  is defined by  $\overline{f}(t) = \overline{f(t)}$ ,  $t \in \Omega$ .  
Other examples:

③ For  $K: [a, b] \times [a, b] \rightarrow \mathbb{K}$  continuous,  $H = L^2([a, b])$  and  
 $T_K: H \rightarrow H$ ,  $(T_K f)(s) = \int_a^b K(s, t) f(t) dt$ , let  $f, g \in C[a, b]$ .

Then  $\langle T_K f, g \rangle = \int_a^b T_K f(s) \overline{g(s)} ds = \int_a^b \left( \int_a^b K(s, t) f(t) dt \right) \overline{g(s)} ds$   
 by Fubini's theorem  $= \int_a^b f(t) \left( \int_a^b \overline{K(s, t)} g(s) ds \right) dt = \langle f, T_K^* g \rangle$ .

We have  $\boxed{(T_K^* g)(t) = \int_a^b \overline{K(s, t)} g(s) ds}$  for  $g \in C[a, b]$ . Extend to get  $H$   
 by continuity.

For boundedness of  $T_K$  on  $C[a, b]$ , note: for  $s \in [a, b]$ ,  $f \in C[a, b]$ ,  
 $\|T_K f\|_2^2 = \int_a^b |T_K f(s)|^2 ds \leq \int_a^b \left| \int_a^b K(s, t) f(t) dt \right|^2 ds$  Cauchy-Schwarz  
 $\leq \int_a^b \left( \int_a^b |K(s, t)|^2 dt \right) \|f\|_2^2 ds = \|f\|_2^2 \|K\|_2^2$ , thus  $\|T_K\| \leq \|K\|_2$  so  $T_K \in B(H)$

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②

Remark: The adjoint operator exists in the more general case where  $T \in B(\mathcal{H}, \mathcal{K})$ , for  $\mathcal{H}, \mathcal{K}$  Hilbert spaces. There is a unique

$T^* \in B(\mathcal{K}, \mathcal{H})$  s.t.  $\forall x \in \mathcal{H}, \forall y \in \mathcal{K}$ :

$$\langle Tx, y \rangle_{\mathcal{K}} = \langle x, T^*y \rangle_{\mathcal{H}}, \text{ see } \underline{[R, Thm. 13.1]}.$$

Proposition Given  $T \in B(\mathcal{H})$ , we have:

(1)  $\ker T^* = T(\mathcal{H})^\perp$ .

(2)  $\overline{T(\mathcal{H})} = (\ker T^*)^\perp$

Proof: (1) From  $\langle Tx, y \rangle = \langle x, T^*y \rangle, \forall x, y \in \mathcal{H}$ , we see that  $\ker T^* \subseteq T(\mathcal{H})^\perp$ . Let  $y \in T(\mathcal{H})^\perp$ , then  $\langle x, T^*y \rangle = 0, \forall x \in \mathcal{H}$ , so  $T^*y \in \mathcal{H}^\perp = \{0\}$ . Thus,  $y \in \ker T^*$ .

(2) follows from (1) using that  $(T(\mathcal{H})^\perp)^\perp = \overline{T(\mathcal{H})}$ .

Definition If  $\mathcal{H}$  is a Hilbert space and  $T \in B(\mathcal{H})$ , then  $T$  is self-adjoint if  $T = T^*$ . Abbreviation:  $T$  is s.a.

Convention for  $T$  s.a.  $\begin{cases} \mathbb{K} = \mathbb{C}, \text{ say } T \text{ is Hermitian.} \\ \mathbb{K} = \mathbb{R}, \text{ say } T \text{ is symmetric.} \end{cases}$

✓ Corollary For  $T \in B(\mathcal{H})$  we have that  $T(\mathcal{H})$  is dense in  $\mathcal{H}$  if and only if  $T^*$  is injective. (i.e.  $T^* \neq 0$ )

Proof: By previous,  $\overline{T(\mathcal{H})} = \mathcal{H} \Leftrightarrow (\ker T^*)^\perp = \mathcal{H} \Leftrightarrow \ker T^* = \mathcal{H}^\perp = \{0\}$

↑  
 $((\ker T^*)^\perp)^\perp = \ker T^*$  since closed.

But  $T^*$  is linear, so  $\ker T^* = 0 \Leftrightarrow T^*$  is injective (Verify this!).

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③

Example: The right shift (unilateral) shift

Let  $\mathcal{H}$  sp. with basis  $B = \{u_j\}_{j \in \mathbb{N}}$ . Define  $S: \mathcal{H} \rightarrow \mathcal{H}$  (with respect to  $B$ ) by  $Su_k \stackrel{(1)}{=} u_{k+1}$  for all  $k \in \mathbb{N}$ .

Claim: There exists  $S \in \mathcal{B}(\mathcal{H})$  determined by (1).

Indeed, let  $x \in \mathcal{H}$ ,  $x = \sum_{j=1}^{\infty} \langle x, u_j \rangle u_j$ .

[Or,  $S_r: (x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, x_3, \dots)$  on  $\ell^2(\mathbb{N})$ , [R, Ex. 11.7]]

Want  $Sx \stackrel{(2)}{=} \sum_{j=1}^{\infty} \langle x, u_j \rangle u_{j+1} = \sum_{j=2}^{\infty} \langle x, u_{j-1} \rangle u_j$ . But this converges,

since  $\sum_{j=2}^{\infty} |\langle x, u_{j-1} \rangle|^2 = \sum_{j=1}^{\infty} |\langle x, u_j \rangle|^2 = \|x\|^2 < \infty$ . So  $Sx$  is

meaningful. We see that  $\|Sx\| = \|x\|, \forall x \in \mathcal{H}$ . Thus  $\|S\| = 1$ .

Also,  $S$  given by (2) is linear (verify this!).

Determine  $S^*: \mathcal{H} \rightarrow \mathcal{H}$ . For  $x \in \mathcal{H}$ ,

$$\langle Sx, u_1 \rangle = \left\langle \sum_{j=2}^{\infty} \langle x, u_{j-1} \rangle u_j, u_1 \right\rangle = 0, \text{ so } \langle x, S^* u_1 \rangle = 0, \forall x.$$

In all,  $S^* u_1 = 0$ .

$$\begin{aligned} \text{For } k \geq 2, \langle Sx, u_k \rangle &= \left\langle \sum_{j=2}^{\infty} \langle x, u_{j-1} \rangle u_j, u_k \right\rangle = \sum_{j=2}^{\infty} \langle x, u_{j-1} \rangle \langle u_j, u_k \rangle \\ &= \langle x, u_{k-1} \rangle = \langle x, S^* u_k \rangle, \forall x. \end{aligned}$$

Thus  $S^* u_k = \begin{cases} 0 & \text{if } k=1 \\ u_{k-1} & \text{if } k \geq 2. \end{cases}$

On  $\ell^2(\mathbb{N})$  with  $\{e_j\}_{j \in \mathbb{N}}$ ,  $S_r^* = S_l^*$ ,  $S_l: (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, \dots)$

Def: Self-adjoint operator is  $T \in B(H)$  s.t.  $T = T^*$ .

1) For  $A = [a_{jk}] \in M_n(K)$ ,  $T_A \in B(K^n)$ ,  $T_A x = Ax$ .

So  $T$  s.a.  $\Leftrightarrow T = T^* \Leftrightarrow A^t = A \Leftrightarrow \overline{a_{kj}} = a_{jk}, \forall k, j$ .

When  $K = \mathbb{R}$ :  $a_{kj} = a_{jk}, \forall k, j$ . Call  $T$  symmetric.

2)  $D u_k = \lambda_k u_k$ , diagonal operator on  $H$  with  $\{u_j\}$  o.n.b.  
 $\lambda \in \ell^\infty(H)$ .

$D$  s.a.  $\Leftrightarrow \lambda_k = \overline{\lambda_k}; \forall k \Leftrightarrow \lambda_k \in \mathbb{R}, \forall k$

3) Note:  $S$  shift not self-adjoint.

4)  $M_f$  s.a.  $\Leftrightarrow f = \overline{f} \in \mathbb{R}$  on  $H = L^2(\Sigma, A, \mu)$

5)  $T_K$  for  $K: C([a, b]^2) \rightarrow K$  s.a. if  $K(s, t) = \overline{K(t, s)}$   
 $\forall s, t$ .

6) Orthogonal projections.

Let  $M \subseteq H$  be a closed subspace. What is  $P_M$ ?

For  $x \in H$ ,  $P_M x - x \in \ker P_M = M^\perp$ , so  $P_M x = x + (P_M x - x)$ .  
 $\uparrow$  Since  $P_M^2 = P_M$ .

$$0 = \langle P_M x - x, P_M y \rangle = \langle P_M x, P_M y \rangle - \langle x, P_M y \rangle.$$

$$\langle x, P_M y \rangle = \langle P_M x, P_M y - y \rangle = \langle P_M x, P_M y \rangle - \langle P_M x, y \rangle.$$

$$\text{Hence } \langle P_M x, y \rangle = \langle x, P_M y \rangle (= \langle P_M x, P_M y \rangle), \forall x, y \in H.$$

$$\text{By uniqueness of adjoint, } P_M^* = P_M.$$

Some algebraic constructions that produce  
s.a. operators

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⑤

Proposition [B, 2.5.7]. Let  $T \in B(\mathcal{H})$ . Then  $T + T^*$ ,  $T^*T$  and  $TT^*$  are self-adjoint.

If  $K = \mathbb{C}$ , then  $\frac{1}{i}(T - T^*)$  is self-adjoint.

Proof: Use the basic properties, e.g.  $(T^*T)^* = T^*(T^*)^* = T^*T$ .

For  $K = \mathbb{C}$ ,  $[\frac{1}{i}(T - T^*)]^* = \frac{1}{i}(T^* - T) = -\frac{1}{i}(T^* - T) = \frac{1}{i}(T - T^*)$

[Think of  $\frac{z - \bar{z}}{i} = \frac{(a+ib) - (a-ib)}{i} = \frac{2ib}{i} = 2b \in \mathbb{R}$ ]

Corollary Let  $\mathcal{H}$  be a complex Hilbert space and  $T \in B(\mathcal{H})$ .

Set  $\operatorname{Re}(T) = \frac{1}{2}(T + T^*)$ ,  $\operatorname{Im}(T) = \frac{1}{2i}(T - T^*)$ . We have:

(i)  $\operatorname{Re}(T)$ ,  $\operatorname{Im}(T)$  are s.a.

(ii)  $T = \operatorname{Re} T + i \operatorname{Im} T$ . (Simple verification)

Proof: Immediate from previous proposition.