

The numerical radius and numerical range of an operator

For $T \in B(H)$, the numerical range of T is

$$W_T = \{ \langle Tx, x \rangle \mid x \in H, \|x\| = 1 \} \subseteq \mathbb{K}.$$

[We always assume $H \neq \{0\}$]

The numerical radius of T is:

$$N_T = \sup \{ |\lambda| \mid \lambda \in W_T \} = \sup \{ |\langle Tx, x \rangle| \mid x \in H, \|x\| = 1 \}.$$

Theorem [B, 2.5.9] If T is self-adjoint in $B(H)$, then

$$\|T\| = N_T. \quad (\text{true for any } T \in B(H)), \text{ is that}$$

Proof: Easy inequality $\forall: |\langle Tx, x \rangle| \leq \|Tx\| \cdot \|x\| \leq \|T\| \cdot \|x\|^2$ by Cauchy-Schwartz, so $N_T \leq \|T\|$. Need $N_T \geq \|T\|$. Here we use that $T = T^*$.

Note: For $x \in H$, $\langle Tx, x \rangle = \langle x, T^*x \rangle = \langle x, Tx \rangle = \overline{\langle Tx, x \rangle}$,
so $\langle Tx, x \rangle \in \mathbb{R}$. (else it is clear)

Note: For $\forall v \in H$, $|\langle Tv, v \rangle| \leq N_T \|v\|^2$. Indeed, if $v \neq 0$, then

this means $|\langle T(\frac{v}{\|v\|}), \frac{v}{\|v\|} \rangle| \leq N_T$, which is clear by def. of N_T .

We need to show that $\forall x, \|x\| = 1, \|Tx\| \leq N_T$.

Let $x \in H$ with $\|x\| = 1$. If $T(x) = 0$, then clear. Assume $T(x) \neq 0$.

Then $\|Tx\| = \langle T(\frac{x}{\|Tx\|}), Tx \rangle = \langle Tx, T(\frac{x}{\|Tx\|}) \rangle = \langle Tx, y \rangle$, using that

$\|Tx\|^2 = \langle Tx, Tx \rangle$. We also have $\langle y, Tx \rangle = \langle Ty, x \rangle$.

In all, $\|Tx\| = \langle Tx, y \rangle = \langle Ty, x \rangle$.

$$\|Tx\| = \frac{1}{2}(\langle Ty, x \rangle + \langle Tx, y \rangle)$$

Idea: add terms to $\langle Tu, u \rangle$ get summands $\langle Tu, u \rangle$
cross-terms

$$\langle T(x+y), x+y \rangle = \langle Tx, x \rangle + \langle Tx, y \rangle + \langle Ty, x \rangle + \langle Ty, y \rangle$$

$$\langle T(x-y), x-y \rangle = \langle Tx, x \rangle - \langle Tx, y \rangle - \langle Ty, x \rangle + \langle Ty, y \rangle$$

$$\langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle = 2(\langle Tx, y \rangle + \langle Ty, x \rangle)$$

$$\Rightarrow \|Tx\| = \frac{1}{4} (\langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle)$$

$$\leq \frac{1}{4} N_T (\|x+y\|^2 + \|x-y\|^2) \text{ by earlier}$$

$$\leq \frac{1}{4} N_T (2\|x\|^2 + 2\|y\|^2) \text{ by the parallelogram law}$$

$$\leq \frac{1}{2} N_T (1 + \|y\|^2) \leq N_T; \text{ this implies } \|T\| \leq N_T.$$

Lemma Suppose that $S: H \rightarrow H$ is linear and $\|Sx\| = \|x\|$, $\forall x \in H$. □

Then, $\langle Sx, Sy \rangle = \langle x, y \rangle$, $\forall x, y \in H$.

Proof: By the polarization identity,

$$\langle Sx, Sy \rangle = \frac{1}{4} \sum_{k=0}^3 i^k \|Sx + i^k Sy\|^2$$

$$= \frac{1}{4} (\|Sx + Sy\|^2 + i\|Sx + iSy\|^2 - \|Sx - Sy\|^2 - i\|Sx - iSy\|^2)$$

$$= \frac{1}{4} (\|S(x+y)\|^2 + i\|S(x+iy)\|^2 - \|S(x-y)\|^2 - i\|S(x-iy)\|^2)$$

$$= \frac{1}{4} (\|x+y\|^2 + i\|x+iy\|^2 - \|x-y\|^2 - i\|x-iy\|^2)$$

$$= \langle x, y \rangle, \forall x, y \in H.$$

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Unitary operators (See [B, 2.6])

Definition: A map $T: X \rightarrow X$ is isometric if $\|Tx\| = \|x\|$, $\forall x \in X$
(Here X is a normed space) (or is an isometry)

Let H be a Hilbert space. A map $T: H \rightarrow H$ preserves the i.p. if $\langle Tx_1, Tx_2 \rangle = \langle x_1, x_2 \rangle$ for all $x_1, x_2 \in H$.

Proposition ① For a map $S: H \rightarrow H$, the following are equivalent:

1. (i) $S \in B(H)$ and $S^*S = I_H$, identity on H .

1. (ii) S is linear and preserves the inner product.

1. (iii) S is isometric.

See previous lemma.

Proof: (i) $\Rightarrow$ (ii) $S \in B(H)$ is already linear. For $x_1, x_2 \in H$,

$$\langle Sx_1, Sx_2 \rangle = \langle x_1, S^*Sx_2 \rangle = \langle x_1, x_2 \rangle, \text{ as claimed.}$$

(ii) $\Rightarrow$ (iii). Only need to show $\|Sx\| = \|x\|$, $\forall x \in H$. But this is immediate from i.p. preserving.

(iii) $\Rightarrow$ (i) From $\|Sx\| = \|x\|$, $\forall x$ we get $S \in B(H)$ with $\|S\| = 1$.

Clearly, $I - S^*S$ is s.a. Since

$$\langle (I - S^*S)x, x \rangle = \langle x, x \rangle - \langle S^*Sx, x \rangle = 0, \forall x, \text{ it follows}$$

that $N_{I - S^*S} = H$. By earlier theorem, $\|I - S^*S\| = 0$.

Necessarily we get $I - S^*S = 0$, so $S^*S = I$.

Remark: If $\dim(H) < \infty$, then $S: H \rightarrow H$ linear and isometric is automatically bijective with $S^{-1} \in B(H)$.

Why? Note that $\|Sx\| = \|x\|$, $\forall x$ implies $\ker S = \{0\}$. Thus, S is

injective, so it is surjective ($\dim(S(H)) = \dim H - \dim(\ker S)$).

Hence $S^{-1}: H \rightarrow H$ exists (and is unique) $\Rightarrow S^{-1} = S^* \in B(H)$. (since S^* is a right inverse.)

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Remark: When $\dim(H) = \infty$, S can be an isometry but not be surjective (so is not invertible). For $H = \ell^2(\mathbb{N})$ (4)

and $S: \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$, S right shift, with $u_j = e_j$, $j \in \mathbb{N}$ o.n. basis.

Thus, $Sx = \sum_{j=2}^{\infty} \langle x, u_{j-1} \rangle u_j$ for $x \in \ell^2(\mathbb{N})$ (or H with o.n.b. $\{u_j\}_{j \in \mathbb{N}}$)

Note that $u_1 \notin \text{Ran}(S)$, so S not surjective.

But, $S^*S = I_H$. Indeed, $\forall x \in H$, S^* continuous,

$$\begin{aligned} S^*Sx &= S^*\left(\sum_{j=2}^{\infty} \langle x, u_{j-1} \rangle u_j\right) \stackrel{\downarrow}{=} \sum_{j=2}^{\infty} \langle x, u_{j-1} \rangle S^*u_j \\ &= \sum_{j=2}^{\infty} \langle x, u_{j-1} \rangle u_{j-1} = \sum_{j=1}^{\infty} \langle x, u_j \rangle u_j = x. \end{aligned}$$

Definition For H Hilbert space over $\mathbb{K}$, an operator $U \in B(H)$ is unitary if $U^*U = UU^* = I_H$.

More generally, if H, K are Hilbert spaces and $U \in B(H, K)$, then U is unitary if $U^*U = I_H$ and $UU^* = I_K$.

Note: U unitary $\Rightarrow U$ invertible in the sense that U bijective and $U^{-1} \in B(H)$. Simply note that $U^{-1} = U^*$.

The previous prop. takes the form:

Prop 2 For $U: H \rightarrow H$, TFAE:

2.(i) U is a unitary.

2.(ii) U is bijective, linear, and inner-product preserving.

2.(iii) U is a surjective linear isometry.

Proof: 2.(i) $\Rightarrow$ 2.(ii) follows from 1.(i) $\Rightarrow$ 1.(ii) and the fact that a unitary is bijective.

2(ii) $\Rightarrow$ 2(iii) Follows immediately from 1(ii) $\Rightarrow$ 1(iii). 9/09/2025 ⑤

2(iii) $\Rightarrow$ 2(i) From 1(iii) $\Rightarrow$ 1(i) we know that $U^*U = I_H$. This says

that U^* is a left inverse for U . But U is bijective, since injectivity follows because it is isometric. So $U^{-1}U = I_H = UU^{-1}$, and then necessarily $U^{-1} = U^*$ ($U^* = U^*I_H = U^*(U U^{-1}) = (U^*U)U^{-1} = U^{-1}$).

Example ① H with o.n.b. $\{u_j\}_{j \in \mathbb{N}}$, $\{\lambda_j\}$, $\ell^\infty(\mathbb{N})$.

$$U: H \rightarrow H, \quad Ux = \sum_{j=1}^{\infty} \lambda_j \langle x, u_j \rangle u_j, \quad U^*_x = \sum_{j=1}^{\infty} \bar{\lambda}_j \langle x, u_j \rangle u_j$$

$$\text{Check: } UU^*_x = \sum_{j=1}^{\infty} \lambda_j \bar{\lambda}_j \langle x, u_j \rangle u_j = UU^*_x, \quad \forall x.$$

So, U is a unitary provided that $|\lambda_j| = 1$ for all $j \in \mathbb{N}$.

Example ② $H = L^2(\Omega, \mathcal{A}, \mu)$, $f \in L^\infty$.

$$M_f g = fg; \quad \text{Know } M_f^* = M_{\bar{f}}.$$

It follows that: $M_f M_{\bar{f}} = I = M_{\bar{f}} M_f$ if $|f|^2 = 1$ μ -a.e.

Example ③ The two-sided (or bilateral shift)

$$\text{Consider } \ell^2(\mathbb{Z}) = \left\{ \xi = \{\xi_j\}_{j \in \mathbb{Z}} \mid \sum_{j=-\infty}^{\infty} |\xi_j|^2 < \infty \right\}.$$

Define $U: \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ by $(U\xi)_j = \xi_{j-1}$ for all $\xi \in \ell^2(\mathbb{Z})$, $j \in \mathbb{Z}$ (right shift)

Verify that U^* is determined as $U^*: \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$, $(U^*\xi)_j = \xi_{j+1}$.

So $(U^*U\xi)_j = \xi_{j+1} = \xi_j, \forall j$. Similarly, $UU^* = I$. Got unitary

$$\xi = \begin{pmatrix} \dots & \xi_{-2} & \xi_{-1} & \xi_0 & \xi_1 & \xi_2 & \dots \\ \dots & -1 & 0 & 1 & 0 & 0 & \dots \end{pmatrix}$$

Alternatively, note that U is linear, surjective and isometric (satisfies 2(i)). For $\xi \in \mathcal{H}$, $\|\xi\|_2^2 = \sum_{j=-\infty}^{\infty} |\xi_j|^2 = \sum_{j \in \mathbb{Z}} |\xi_j|^2$ 9/09/2025 (6)

$$\|U\xi\|^2 = \sum_{j \in \mathbb{Z}} |(U\xi)_j|^2 = \sum_{j \in \mathbb{Z}} |\xi_{j-1}|^2 = \sum_{j \in \mathbb{Z}} |\xi_j|^2 = \|\xi\|^2.$$

U surjective: for $\eta = (\dots, \eta_{-1}, \eta_0, \eta_1, \dots)$

let $\xi = (\dots, \eta_0, \eta_1, \eta_2, \dots)$. Then $U\xi = \eta$.

Def: Two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ are isomorphic when there exists a unitary $U: \mathcal{H} \rightarrow \mathcal{K}$.

Theorem Suppose that $\mathcal{H}$ is a complex Hilbert space and

B is an o.n.b for $\mathcal{H}$, then $\mathcal{H}$ and $\ell^2(B)$ are isomorphic.

Recall: B o.n.b basis means B is an o.n. set in $\mathcal{H}$ such that $\text{span}(B)$ is dense in $\mathcal{H}$.

$$\ell^2(B) = \left\{ \xi: B \rightarrow \mathbb{C} \mid \sum_{u \in B} |\xi(u)|^2 < \infty \right\}$$

as summable series with positive terms.

Proof: For $x \in \mathcal{H}$, define its "Fourier coefficient"

$$\hat{x}: B \rightarrow \mathbb{C}, \quad \hat{x}(u) = \langle x, u \rangle, \quad \forall u \in B.$$

Parseval's ineq $\Rightarrow \hat{x} \in \ell^2(B)$ with $\|\hat{x}\| = \|x\|$, $\forall x$.

Set $U: \mathcal{H} \rightarrow \ell^2(B)$, $U(x) = \hat{x}$, for all $x \in \mathcal{H}$. Then U is

linear and isometric. We also have U surjective; let $\xi \in \ell^2(B)$,

so $\sum_{u \in B} |\xi(u)|^2 < \infty$. We know that this sum is at most countable,

say $B_\xi = \{u_j\}_{j \in \mathbb{N}} = \{u \in B \mid \xi(u) \neq 0\}$. Then $x := \sum_{j \in \mathbb{N}} \xi(u_j) u_j \in \mathcal{H}$.
Verify $Ux = \xi$. $\Rightarrow U$ unitary.