

Compact Linear OperatorsSome preliminaries:

Given (X, d) metric space and $\{x_n\}_{n \in \mathbb{N}}$ a sequence in X , by a subsequence of $\{x_n\}$ we mean a sequence of the form $\{x_{n_k}\}_{k \in \mathbb{N}}$ where $n_1 < n_2 < n_3 < \dots$.

(Note: each x_{n_k} is an element of the original sequence).

Remark: (1) A subsequence of $\{x_n\}_{n \in \mathbb{N}}$ has form $\{x_{f(k)}\}$ where $f: \mathbb{N} \rightarrow \mathbb{N}$ is monotone increasing.

(2) If $\{x_{n_k}\}_{k \in \mathbb{N}}$ is a subsequence of $\{x_n\}_n$, then $n_k \geq k, \forall k \in \mathbb{N}$.

Why? Since $n_1 \in \mathbb{N}$ and $n_2 > n_1$ we have $n_1 \geq 1$ and $n_2 \geq 2$. By induction, $n_k \geq k$ (since $n_k > n_{k-1} \geq k-1$ for all $k \geq 3$).

Def. Let X and Y be normed spaces and $T: X \rightarrow Y$ linear.

We say T is compact if for every bounded sequence $\{x_n\}_n \subseteq X$, the sequence $\{T(x_n)\}_{n \in \mathbb{N}}$ in Y has a convergent subsequence.

So T compact if $\forall \{x_n\}_n \subseteq X$ bounded, $\exists \{x_{n_k}\}_{k \in \mathbb{N}}$ s.t. $\{T(x_{n_k})\}_k$ is convergent in Y .

Theorem: If a linear operator $T: X \rightarrow Y$ is compact, then it is bounded.

Proof: Assume T is compact but not bounded. For every $M > 0$, we have $\sup \{\|Tx\| : x \in X, \|x\| \leq 1\} \geq M$. For each $n \geq 1$ choose $x_n \in X$ with $\|x_n\| \leq 1$ and $\|Tx_n\| \geq n$. Then there is $\{x_{n_k}\}$ subseq. such that $\{Tx_{n_k}\}_k$ converges. But $\{Tx_{n_k}\}_k$ not bounded, a falsehood. $\square$

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A characterization of compactness

Theorem Let X, Y be normed spaces and $T: X \rightarrow Y$ linear.

Then T is compact if and only if $\forall A \subseteq X$ bounded, the set $\overline{T(A)}$ is compact. (This says that $T(A)$ is precompact).

Proof: " $\Rightarrow$ ".

Assume T compact, let $A \subseteq X$ bounded, so $\exists M > 0$ s.t. $\|a\| \leq M$ for every $a \in A$. Take $\{y_n\} \subseteq T(A)$. Need a convergent subsequence.

For each $n \geq 1$ $\exists a_n \in A$: $\|y_n - T(a_n)\| < \frac{1}{n}$. Since A is bdd, $\{a_n\}$ is bounded. By compactness of T , there is $\{a_{n_k}\}_k$ subseq. s.t. $\{T(a_{n_k})\}$ converges. Let $y = \lim_k T(a_{n_k}) \in Y$.

Claim: $\lim_k y_{n_k} = y$. This will show that $\overline{T(A)}$ is compact.

$$\text{But } \|y_{n_k} - y\| \leq \underbrace{\|y_{n_k} - T(a_{n_k})\|}_{\leq \frac{1}{n_k} \leq \frac{1}{k}} + \|T(a_{n_k}) - y\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $k \geq k_\varepsilon$,
for some k_ε .

" $\Leftarrow$ " Let $\{x_n\}_n \subseteq X$ be bounded. Let $A = \{x_n \mid n \in \mathbb{N}\}$.

Then $\{T(x_n)\}_n \subseteq T(A) \subseteq \overline{T(A)}$ has a convergent subsequence, so of form $\{T(x_{n_k})\}_k$. In particular, $\{x_{n_k}\}_k$ is a subsequence of $\{x_n\}$ and $\{T(x_{n_k})\}_k$ converges. $\square$

Corollary Let $T: X \rightarrow Y$ be linear. Then T is compact iff

$\{T(x) \mid \|x\| \leq 1\}$ has compact closure in Y (is precompact).

↑ The image in Y of the closed unit ball $\{x \in X \mid \|x\| \leq 1\}$

" $\Rightarrow$ " Clear. For " $\Leftarrow$ " Let $A \subseteq X$ bounded with $\|x\| \leq M, \forall x \in A$, with $M > 0$.

Then $\{T(M^{-1}x) \mid x \in A\}$ is compact. But then $\{T(x) \mid x \in A\}$ is compact.

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(3)

Notation: For X, Y normed spaces, let

$K(X, Y) = \{T: X \rightarrow Y \mid T \text{ compact linear}\}$. Write $K(X)$ when $Y = X$.

Algebraic properties of $K(X, Y)$

Proposition Let X, Y, Z normed spaces.

(a) If $S, T \in K(X, Y)$ and $\alpha, \beta \in \mathbb{K}$, then $\alpha S + \beta T \in K(X, Y)$.

Skip in Class (b) If $S \in K(X, Y)$, $R \in K(Y, Z)$ then $RS \in K(X, Z)$ if at least one of S and R is compact.

Proof:

(a) Show $S+T$ compact. Let $\{x_n\}_n \subseteq X$ bounded.

Choose $\{x_{n_k}\}_k$ s.t. $\{S(x_{n_k})\}_k$ convergent,

then choose a subseq $\{x_{n_{k_l}}\}_l$ of $\{x_{n_k}\}_k$ s.t. $\{T(x_{n_{k_l}})\}_l$ conv.

It follows that $\{(S+T)(x_{n_{k_l}})\}_l$ is convergent.

Read this (b) Let $\{x_n\} \subseteq X$ be bounded, with $M > 0: \|x_n\| \leq M$ for every $n \geq 1$.

Case: S compact. Find subseq. $\{x_{n_k}\}_k$ s.t. $\{S(x_{n_k})\}_k$ converges.

Since R is bounded, also $\{R(S(x_{n_k}))\}_k$ converges, so RS compact.

Case: R compact. Note that $\{S(x_n)\}_n$ is bounded by $\|S\| \cdot M$.

Then there exist a subseq. $\{S(x_{n_k})\}_k$ s.t. $\{R(S(x_{n_k}))\}_k$ converges.

But $\{x_{n_k}\}_k$ is a subsequence of $\{x_n\}_n$, so RS compact. $\square$

Def: A linear operator $T: X \rightarrow Y$ between normed spaces has finite rank if $T(X)$ is finite dimensional.

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Proposition Let X, Y normed spaces and $T \in B(X, Y)$ ④

(a) If T has finite rank, then $T \in K(X, Y)$.

(b) If X or Y is finite dimensional, then $T \in K(X, Y)$.

Proof: (a) Let $T(X)$ be finite dimensional. Since $T(X)$ is a finite dimensional subspace of Y , it is isomorphic to $\mathbb{R}^n$ or $\mathbb{C}^n$.

Then $\{Tx \mid \|x\| \leq 1\}$ is a bounded subset (bounded by $\|T\|$) of $T(X)$.

So $\overline{\{Tx \mid \|x\| \leq 1\}}$ is closed and bounded in $T(X)$.

Then $\overline{\{Tx \mid \|x\| \leq 1\}}$ is compact by a result in Lecture ①.

(b) If Y finite dim, then $T(X)$ fin. dim so $T \in K(X, Y)$ by (a).

If X finite dim., then $\dim T(X) \leq \dim X$ so T has finite rank.

(a) Alternatively, let $\{x_n\} \subseteq X$ be bounded by $M > 0$. Then

$$\|Tx_n\| \leq \|T\| \cdot \|x_n\| \leq \|T\| \cdot M, \text{ so } \{Tx_n \mid n \in \mathbb{N}\} \subseteq \{y \in Y \mid \|y\| \leq \|T\| \cdot M\} = B_{\|T\| \cdot M} \\ \text{closed ball.}$$

Since $\{Tx_n\} \subseteq T(X) \cap B$, a compact set (being closed and bounded), there exists a subsequence $\{Tx_{n_k}\}_k$ which converges.

Examples of compact operators. (using the proposition above)

(1) Let $X = Y = \mathcal{H}$, Hilbert space ($\neq \{0\}$). Let $y \in \mathcal{H}, y \neq 0$.

Define $T: \mathcal{H} \rightarrow \mathcal{H}$, $Tz = \langle z, y \rangle y = T_{yy}(z)$. Then T is a rank-one operator, thus is compact, i.e. $T \in K(\mathcal{H})$.

Note: if $\|y\| = 1$ then $T = P_{\{y\}}$.

(2) $T: \mathcal{H} \rightarrow \mathcal{H}$, $T(w) = \sum_{i=1}^n \sum_{j=1}^m \langle z, w_i \rangle y_j$, $\{w_i\}_{1 \leq i \leq n}, \{y_j\}_{1 \leq j \leq m} \subseteq \mathcal{H}$.

Then $T = \sum_{i=1}^n \sum_{j=1}^m T_{w_i y_j}$ so it is compact as a sum of compact operators.

(3) $X = C[0, 1, \mathbb{R}]$ with $\|\cdot\|_\infty$, $T: X \rightarrow X$, $(Tg)(s) = \int_0^1 \sin(s-t)g(t)dt$, $s \in [0, 1]$.
Verify: $\dim(T(X)) = 2$, $\|Tg\|_\infty \leq \|g\|_\infty$, $\forall g \in X$. [See [B, ex. 4.1.5.]]

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Let X be a normed space and Y a Banach space. If $\{T_k\}_{k \in \mathbb{N}}$ is a sequence in $\mathcal{K}(X, Y)$ that converges to some $T \in \mathcal{B}(X, Y)$, then $T \in \mathcal{K}(X, Y)$. As a consequence, $\mathcal{K}(X, Y)$ is a closed subspace of $\mathcal{B}(X, Y)$ for the operator norm. Thus, $\mathcal{K}(X, Y)$ is complete.

Proof: Let $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ be bounded. We need a convergent subsequence of $\{T(x_n)\}_{n \in \mathbb{N}}$.

Two main steps: (I) - A "diagonal" argument to find $\{x_n\}_1^\infty$.

(II) An $\frac{\varepsilon}{3}$ -argument to show that $\{T(x_n)\}_n$ is a Cauchy sequence, therefore convergent.

(I) Since T_1 compact, find $\{x_{m_i, l}\}_{l \in \mathbb{N}}$ subseq. of $\{x_n\}_n$ s.t. $\{T_1(x_{m_i, l})\}_{l \in \mathbb{N}}^{\text{conv}}$

Since T_2 compact, find $\{x_{n_2, l}\} \in \mathbb{N}$ subseq. of $\{x_{n_1, l}\}_l$ s.t. $\{T_2(x_{n_2, l})\}_l$ conv.

Note: also $\{T_1(x_{n_2, l})\}_l$ conv.

Continue:

since T_k compact, find $\{x_{n_{k,l}}\}_{l \in \mathbb{N}}$ subseq. of $\{x_{n_{k,l}}\}_{l \in \mathbb{N}}$ st.

$\{T_k(x_{n_{k,l}})\}_l$ converges, as do, $\{T_j(x_{n_{k,l}})\}_l$ for all $1 \leq j \leq k$.

If we continue like this, we might get an empty sequence. So, form

$n_l := n_{l,l}$ for $l \geq 1$. Claim: This is a subsequence.

$$n_{1,1} < n_{1,2} < n_{1,3} < \dots$$

$$n_{2,2} \leq n_{2,3} \leq \dots$$

$$\leq n_{3,3} < \dots \quad \text{So } n_\ell < n_{\ell+1}, \forall \ell \geq 1.$$

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Thus $\{x_{n_l}\}_l$ is a subsequence of $\{x_{n_1, l}\}_l, \{x_{n_2, l}\}_l, \dots$. ⑥

We therefore have $\{T_k(x_{n_l})\}_{l \geq k}$ converges, (take $j \leq k = l$ above and write j instead of k) for all $k \geq 1$.

(II) Let $\varepsilon > 0$. Consider

$$\|Tx_{n_l} - Tx_{n_j}\| \leq \underbrace{\|(T - T_K)x_{n_l}\|}_{< \frac{\varepsilon}{3}} + \underbrace{\|T_K(x_{n_l}) - T_K(x_{n_j})\|}_{\frac{\varepsilon}{3}} + \underbrace{\|(T - T_K)x_{n_j}\|}_{< \frac{\varepsilon}{3}}.$$

Want:

Let $M > 0$ s.t. $\|x_n\| \leq M$ for all n .

Let $K \in \mathbb{N}$ s.t. $\|T - T_K\| < \frac{\varepsilon}{3M}$. Then, $\|(T - T_K)x_{n_l}\| \leq \|T - T_K\| \cdot \|x_{n_l}\| < \frac{\varepsilon}{3M} \cdot M = \frac{\varepsilon}{3} \forall l$.

Since $\{T_K(x_{n_l})\}_{l \geq K}$ converges, find $L \in \mathbb{N}$ s.t.

$$l, j \geq L \Rightarrow \|T_K(x_{n_l}) - T_K(x_{n_j})\| < \frac{\varepsilon}{3}.$$

In all; $l, j \geq L \Rightarrow \|Tx_{n_l} - Tx_{n_j}\| < \varepsilon$, so $\{Tx_{n_l}\}_l$ Cauchy, thus convergent. Hence $T \in \mathcal{K}(X, Y)$. $\square$

Let $\mathcal{F}(X, Y) = \{T \in \mathcal{B}(X, Y) : T \text{ has finite-rank}\}$

Corollary If Y is a Banach space, we have

$$\overline{\mathcal{F}(X, Y)} \subseteq \mathcal{K}(X, Y).$$

Proof: combine earlier results.