

Exam MAT4410, autumn 2025
Suggested solution

Problem 1a. Let $x, y \in H$. We have

$$\begin{aligned}\langle T(x+y), x+y \rangle &= \langle Tx + Ty, x+y \rangle = \langle Tx, x \rangle + \langle Tx, y \rangle + \langle Ty, x \rangle + \langle Ty, y \rangle \\ &= \langle x+y, T(x+y) \rangle \\ &= \langle x, Tx \rangle + \langle x, Ty \rangle + \langle y, Tx \rangle + \langle y, Ty \rangle.\end{aligned}$$

After cancelling terms according to the hypothesis, we get the identity

$$(1) \quad \langle Tx, y \rangle + \langle Ty, x \rangle = \langle x, Ty \rangle + \langle y, Tx \rangle,$$

which is valid for arbitrary $x, y \in H$. Similarly, we have

$$\begin{aligned}\langle T(x+iy), x+iy \rangle &= \langle Tx + iTy, x+iy \rangle = \langle Tx, x \rangle - i\langle Tx, y \rangle + i\langle Ty, x \rangle + \langle Ty, y \rangle \\ &= \langle x+iy, T(x+iy) \rangle \\ &= \langle x, Tx \rangle - i\langle x, Ty \rangle + i\langle y, Tx \rangle + \langle y, Ty \rangle.\end{aligned}$$

Once again cancelling terms according to the hypothesis, we obtain the identity

$$(2) \quad \langle Tx, y \rangle - \langle Ty, x \rangle = \langle x, Ty \rangle - \langle y, Tx \rangle,$$

valid for arbitrary x, y in H . By adding up (1) and (2) term by term, we obtain $2\langle Tx, y \rangle = 2\langle x, Ty \rangle$ for arbitrary x, y in H . Since $T \in B(H)$, there exists a unique adjoint operator T^* in $B(H)$ satisfying that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for every $x, y \in H$. Thus we conclude that $T^* = T$, which means exactly that T is self-adjoint.

Problem 1b. Assume that T is positive in $B(H)$ and S is arbitrary in $B(H)$. Let $x \in H$. By positivity of T , we have $\langle T(Sx), Sx \rangle \geq 0$. With $S^* \in B(H)$ being the adjoint of S , we thus have

$$\langle S^*T(Sx), x \rangle = \langle (S^*TS)x, x \rangle \geq 0,$$

which means exactly that S^*TS is positive.

Problem 2. To ease notation, we write ℓ^2 instead of $\ell^2(\mathbb{N})$. For part (2a), let $C = \sup_{n \in \mathbb{N}} |\alpha_n| = \|\{\alpha_n\}\|_1$. Given $\{x_n\} \in \ell^2$, for every $N \in \mathbb{N}$ we have

$$\sum_{n=1}^N |\alpha_n x_n|^2 \leq \sum_{n=1}^N C^2 |x_n|^2 \leq C^2 \|\{x_n\}\|_2^2.$$

Letting $N \rightarrow \infty$ shows that $\{\alpha_n x_n\} \in \ell^2$ and, moreover, $\|\{\alpha_n x_n\}\|_2 \leq C \|\{x_n\}\|_2$. This implies that the map T in part 2a is well-defined. For $\{x_n\}, \{y_n\} \in \ell^2$ and $\lambda, \mu \in \mathbb{C}$ we have linearity in each coordinate, namely $\alpha_n(\lambda x_n + \mu y_n) = \lambda \alpha_n x_n + \mu \alpha_n y_n$ for every $n \in \mathbb{N}$, thus the map T is linear. By the first part of the proof, we get $T \in B(\ell^2)$ with $\|T\| \leq C = \sup_{n \in \mathbb{N}} |\alpha_n|$. To show that this inequality is an equality, it suffices to show that for all $n \in \mathbb{N}$ we have $\|T\| \geq |\alpha_n|$. Given $n \in \mathbb{N}$, let $e^{(n)} \in \ell^2$ be the unit vector given by $e_j^{(n)} = \delta_{j,n}$ for all $j \geq 1$. Then

$$\|T\| \geq \|Te^{(n)}\|_2 = \|\alpha_n e^{(n)}\|_2 = |\alpha_n|,$$

as needed.

Problem 2b. let $\varepsilon > 0$. There is $N_1 \in \mathbb{N}$ such that $|\alpha_n| < \varepsilon$ for every $n \geq N_1 + 1$. For each $N \in \mathbb{N}$, define a linear operator $T_N : \ell^2 \rightarrow \ell^2$ by

$$T_N(x_1, x_2, \dots) = (\alpha_1 x_1, \dots, \alpha_N x_N, 0, \dots).$$

This is clearly linear and bounded. Since $T_N(\ell^2) = \text{span}\{e^{(1)}, \dots, e^{(N)}\}$, the operator T_N has finite rank, therefore it is compact. We claim that $\lim_{N \rightarrow \infty} \|T - T_N\| = 0$. This will show that T is the limit of a sequence of finite rank operators and will therefore give the conclusion that T is compact. Given $x = \{x_n\} \in \ell^2$, for every $N \geq N_1$ we have

$$\|(T - T_N)x\|_2^2 = \sum_{n \geq N+1} |\alpha_n|^2 |x_n|^2 < \varepsilon^2 \sum_{n \geq N+1} |x_n|^2 \leq \varepsilon^2 \|x\|_2^2.$$

Taking supremum over x we get that $\|T - T_N\| < \varepsilon$ for every $N \geq N_1$, as claimed.

Problem 2c. Assume that T is compact but $\lim_{n \rightarrow \infty} \alpha_n \neq 0$. There exists $\varepsilon > 0$ such that for each $k \in \mathbb{N}$ we can find α_{n_k} satisfying

$$|\alpha_{n_k}| \geq \varepsilon.$$

Consider the bounded sequence $\{x_k\}_{k \in \mathbb{N}}$ given as $x_k = e^{(n_k)}$ for $k \geq 1$. For $k, m \in \mathbb{N}$ we have

$$\|Tx_k - Tx_m\|_2^2 = |\alpha_k|^2 + |\alpha_m|^2 > 2\varepsilon^2,$$

so $\{Tx_k\}_k$ cannot have any convergent subsequence, a contradiction with compactness of T .

Problem 3a. Given a sequence of unit vectors $\|x_n\|$ in H , with $n \in \mathbb{N}$, there is a convergent subsequence $\{Tx_{n_k}\}$ by compactness. Set $y = \lim_k Tx_{n_k}$ in H . Let $\varepsilon > 0$. There are $N_1 \in \mathbb{N}$ such that $\|Tx_{n_k} - y\| < \varepsilon/2$ and $N_2 \in \mathbb{N}$ with

$$k \geq N_2 \Rightarrow \|Tx_{n_k} - \lambda x_{n_k}\| < \varepsilon/2.$$

Now note that for $k \geq \max\{N_1, N_2\}$ we have

$$\|y - \lambda x_{n_k}\| \leq \|y - Tx_{n_k}\| + \|Tx_{n_k} - \lambda x_{n_k}\| < \varepsilon,$$

so x_{n_k} converges to $(1/\lambda)y$.

Problem 3b. Note that $Ty = \lambda T(\frac{1}{\lambda}y) = \lambda \lim_k Tx_{n_k} = \lambda y$, and since $y \neq 0$ (as the limit of a sequence of unit vectors) we conclude that λ is an eigenvalue for T .

Problem 4a. Given $\{\varphi_n\} \subset \mathcal{Z}$ with $\lim_{n \rightarrow \infty} \|\varphi_n - \varphi\| = 0$ for some $\varphi \in X^*$, let $x \in \ker(T)$. Then $\varphi_n(x) = 0$ for every n , so

$$|\varphi(x)| = |\varphi(x) - \varphi_n(x)| \leq \|\varphi - \varphi_n\| \|x\|.$$

Taking supremum over x and using the assumption shows that $|\varphi(x)| = 0$. Since $x \in X$ was arbitrary, we conclude that $\varphi \in \mathcal{Z}$, as needed.

Problem 4b. We now assume that $T(X)$ is closed in X . Since X is a Banach space, we have that $T(X)$ is a Banach space. Let $\varphi \in \mathcal{Z}$. Now, if $Tx_1 = Tx_2$ for $x_1, x_2 \in X$, then $x_1 - x_2 \in \ker(T)$, so $\varphi(x_1 - x_2) = 0$. Hence, $\varphi(x_1) = \varphi(x_2)$. This means that the map $\psi_0 : T(X) \rightarrow \mathbb{C}$ given by $\psi_0(Tx) = \varphi(x)$ is well-defined. To show linearity, let $y_1, y_2 \in T(X)$, which we may assume of the form $y_1 = T(w_1)$ and $y_2 = T(w_2)$ for some

(non necessarily unique) w_1, w_2 in X . Let $\lambda_1, \lambda_2 \in \mathbb{K}$. Then, using linearity of T and φ , we see that

$$\psi_0(\lambda_1 y_1 + \lambda_2 y_2) = \psi_0(T(\lambda_1 w_1 + \lambda_2 w_2)) = \varphi(\lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 \varphi(w_1) + \lambda_2 \varphi(w_2),$$

which equals $\lambda_1 \psi_0(y_1) + \lambda_2 \psi_0(y_2)$, as needed.

We claim that ψ_0 is continuous at 0. Since ψ_0 is linear, it will follow that $\psi_0 \in (T(X))^*$. Let U be the open unit ball in $\mathbb{K}$, then $\varphi^{-1}(U)$ is open in X by continuity of φ . Moreover, given $x \in \ker(T)$, we have $\varphi(x) = 0$, and $\varphi^{-1}(U)$ is an open neighbourhood of x in X .

Let $T_1 : X \rightarrow T(X)$ be the linear operator given by $T_1(x) = T(x)$ for all $x \in X$. Then T_1 is a bounded linear operator, since $\|T_1\| \leq \|T\|$, and is moreover surjective. By the Open Mapping Theorem, T_1 is an open map. We obtain that $T_1(\varphi^{-1}(U)) = T(\varphi^{-1}(U))$ is an open subset of $T(X)$ containing $0 = Tx$.

If $y \in T(\varphi^{-1}(U))$, then $y = Tx$ where $\varphi(x) \in U$. But $\varphi(x) = \psi_0(Tx) = \psi_0(y)$, so $y \in \psi_0^{-1}(U)$. This shows the inclusion

$$T(\varphi^{-1}(U)) \subset \psi_0^{-1}(U).$$

The other inclusion holds, as well. Indeed, let $y \in \psi_0^{-1}(U)$, thus $y = Tx$ and $\psi_0(y) \in U$. This means that $\varphi(x) \in U$, so $x \in \varphi^{-1}(U)$ and therefore $y \in T(\varphi^{-1}(U))$.

In all, $\psi_0^{-1}(U)$ is open in $T(X)$. A similar proof shows that $\psi_0^{-1}(B_\varepsilon(0))$ is an open set containing 0, and we conclude that ψ_0 is continuous at 0. By linearity, ψ_0 is continuous on $T(X)$.

Alternative proof: Let $Y = \ker(T)$, a closed subspace of X . Since X is a Banach space, so is the quotient space X/Y . Define the linear map $\tilde{T} : X/Y \rightarrow T(X)$ by $\tilde{T}(x+Y) = Tx$. This is well-defined by a similar argument to the one showing that ψ_0 is well-defined. Also, $\tilde{T}$ is linear, bounded, and bijective. By the Bounded Inverse theorem, its inverse is bounded. There is also a well-defined bounded linear map $\tilde{\varphi} : X/Y \rightarrow \mathbb{C}$ given by $\tilde{\varphi}(x+Y) = \varphi(x)$ for $x+Y \in X/Y$. It is immediate that $\psi_0 = \tilde{\varphi} \circ \tilde{T}$, and so ψ_0 is bounded as a composition of bounded maps.

Problem 4c. Since ψ_0 is a bounded linear functional on the subspace $T(X)$ of X , the Hahn-Banach theorem provides an extension $\psi \in X^*$ satisfying $\|\psi\| = \|\psi_0\|$. We claim that $T^\times(\psi) = \varphi$. Given $x \in X$, we have

$$T^\times(\psi)(x) = (\psi \circ T)(x) = \psi(Tx) = \psi_0(Tx) = \varphi(x),$$

as needed. It follows that $\mathcal{Z} \subset T^\times(X^*)$. For the converse inclusion, let $\rho = T^\times \eta \in T^\times(X^*)$. For $x \in \ker(T)$, we have $\rho(x) = T^\times \eta(x) = \eta(Tx) = \eta(0) = 0$, so $\rho \in \mathcal{Z}$.