

Fundamental principles of functional analysis

(1)

- Hahn-Banach theorem
- Open mapping theorem
- Uniform boundedness principle

We already know the Hahn-Banach theorem:
given a normed space X (over $K = \mathbb{R}$ or $K = \mathbb{C}$),
a subspace $Y \subset X$ and a bounded linear
functional $f: Y \rightarrow K$, there is a bounded
linear functional $F: X \rightarrow K$ s.t.

$$F|_Y = f, \quad \|F\| = \|f\|.$$

The other two main theorems deal with
operators between normed spaces.

Proposition

Assume $T: X \rightarrow Y$ is a linear operator
between normed spaces (real or complex).

THEOREM:

- (1) T is continuous;
- (2) T is continuous at 0;
- (3) there is $C \geq 0$ s.t. $\|Tx\| \leq C\|x\| \quad \forall x \in X$.

Proof

(1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) If T is continuous at 0 , then $T^{-1}(\{y \in Y : \|y\| < 1\})$ is a neighbourhood of $0 \in X$, hence it contains $\{x \in X : \|x\| < \delta\}$ for some $\delta > 0$. Then for every $x \in X, x \neq 0$, and ~~$x \neq 0$~~ $0 < \delta' < \delta$ we have $\| \delta' \frac{x}{\|x\|} \| = \delta' < \delta$, so $\|T(\delta' \frac{x}{\|x\|})\| < 1$, that is,

$$\|Tx\| < \frac{1}{\delta'} \|x\|.$$

It follows that $\|Tx\| \leq C\|x\|$ for $C = \frac{1}{\delta}$.

(3) $\Rightarrow$ (1)

If $\|Tx\| \leq C\|x\| \quad \forall x$, then

$$\|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\| \leq C\|x_1 - x_2\|,$$

which proves continuity of T . □

If conditions (1) - (3) are satisfied, then T is called a bounded linear operator.

For such T define

$$\begin{aligned} \|T\| &= \inf \{ C > 0 : \|Tx\| \leq C\|x\| \quad \forall x \} \\ &= \min \{ C \geq 0 : \|Tx\| \leq C\|x\| \quad \forall x \}. \end{aligned}$$

Denote by $L(X, Y)$ the set of bounded linear operators $X \rightarrow Y$. We write $L(X)$ for $L(X, X)$.

(2)

Proposition

For any normed spaces X and Y ,
 $(L(X, Y), \|\cdot\|)$ is a normed space. If Y
 is a Banach space, then $(L(X, Y), \|\cdot\|)$ is
 also a Banach space.

Proof

It is easy to see that $(L(X, Y), \|\cdot\|)$ is a
 normed space. For example, if $S \in L(X, Y)$,
 $T \in L(X, Y)$, then

$$\begin{aligned}\|(S+T)x\| &= \|Sx + Tx\| \leq \|Sx\| + \|Tx\| \\ &\leq \|S\| \cdot \|x\| + \|T\| \cdot \|x\|.\end{aligned}$$

This shows that $S+T \in L(X, Y)$ and

$$\|S+T\| \leq \|S\| + \|T\|.$$

Let us prove the second statement.
 So assume Y is complete and $(T_n)_{n=1}^{\infty}$ is
 a Cauchy sequence in $L(X, Y)$. Then, for
 every $x \in X$, the sequence $(T_n x)_{n=1}^{\infty}$ is
 Cauchy in Y , since

$$\|T_n x - T_m x\| \leq \|T_n - T_m\| \cdot \|x\|.$$

Since Y is complete, we get a well-defined
 map $T: X \rightarrow Y$ st.

$$Tx = \lim_{n \rightarrow \infty} T_n x \quad \forall x \in X.$$

It is easy to see that this is a linear
 operator. For example,

$$\begin{aligned} T(x_1 + x_2) &= \lim_{n \rightarrow \infty} T_n(x_1 + x_2) = \lim_{n \rightarrow \infty} (T_n x_1 + T_n x_2) \\ &= \lim_{n \rightarrow \infty} T_n x_1 + \lim_{n \rightarrow \infty} T_n x_2 = T x_1 + T x_2. \end{aligned}$$

Take $\varepsilon > 0$. Let n_0 be s.t. $\|T_n - T_m\| < \varepsilon$
 $\forall n, m \geq n_0$. Then, $\forall x \in X$, we have

$$\|T_n x - T_m x\| \leq \varepsilon \|x\| \quad \forall n, m \geq n_0.$$

Letting $m \rightarrow \infty$ we conclude that

$$\|T_n x - T x\| \leq \varepsilon \|x\| \quad \forall n \geq n_0.$$

(Recall that $|\|x_1\| - \|x_2\|| \leq \|x_1 - x_2\|$, so that
the map $x \mapsto \|x\|$ is continuous.)

This shows that $T_n - T \in L(X, Y)$ and

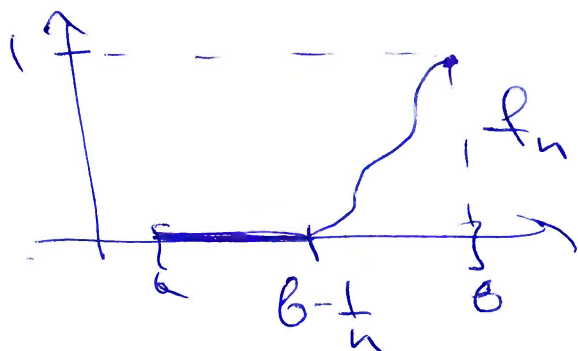
$$\|T_n - T\| \leq \varepsilon. \text{ Hence } T = T_n - (T_n - T) \in L(X, Y).$$

As $\varepsilon > 0$ was arbitrary, we see also that
 $T_n \rightarrow T$ in $L(X, Y)$. $\square$

Examples

1) Consider the operator $T: C^1[a, b] \rightarrow C[a, b]$,
 $Tf = f'$.

Equip both spaces with the sup-norm.
Then T is unbounded. Indeed, for $n \in \mathbb{N}$
consider f_n s.t.



(3)

$$\text{Then } n = \frac{f(b) - f(b - \frac{1}{n})}{b - (b - \frac{1}{n})} = f'_n(c_n)$$

for some $c_n \in [b - \frac{1}{n}, b]$. Hence $\|f'_n\|_\infty \geq n$.

Therefore $\|f_n\|_\infty = 1$, $\|Tf_n\|_\infty \geq n$, so T is unbounded.

On the other hand, if we consider the norm

$$\|f\| \stackrel{\text{def}}{=} \|f\|_\infty + \|f'\|_\infty,$$

then $T: (C[a, b], \|\cdot\|) \rightarrow (C[a, b], \|\cdot\|_\infty)$ is a contraction ($\|T\| \leq 1$).

2) Consider $K \in C([a, b] \times [a, b])$. Consider the integral operator

$$T: C[a, b] \rightarrow C[a, b],$$

$$(Tf)(x) = \int_a^b K(x, y) f(y) dy.$$

Equip $C[a, b]$ with the sup-norm.

$$\text{As } \left| \int_a^b K(x, y) f(y) dy \right| \leq \int_a^b |K(x, y)| |f(y)| dy$$

$$\leq \left(\int_a^b |K(x, y)| dy \right) \|f\|_\infty, \text{ the operator } T$$

$$\text{is bounded, } \|T\| = \sup_{x \in [a, b]} \int_a^b |K(x, y)| dy.$$

Def

Two norms $\|\cdot\|$ and $\|\cdot\|'$ on a vector space X are called equivalent if there exists $C > 0$ s.t.

$$\frac{1}{C} \|x\|' \leq \|x\| \leq C \|x\| \quad \forall x \in X.$$

In other words, the identity maps $(X, \|\cdot\|) \rightarrow (X, \|\cdot\|')$ and $(X, \|\cdot\|') \rightarrow (X, \|\cdot\|)$ are bounded.

Theorem

Any two norms on a f.d. vector space X are equivalent.

Proof

We view X as a real vector space. We may then assume that $X = \mathbb{R}^n$. Consider the Euclidean norm $\|\cdot\|_2$ and another norm $\|\cdot\|$ on $\mathbb{R}^n$. Observe that the map $\|\cdot\|: (\mathbb{R}^n, \|\cdot\|_2) \rightarrow \mathbb{R}$ is continuous. Indeed, let $e_1, \dots, e_n$ be the standard basis in $\mathbb{R}^n$, $C = \max_{1 \leq i \leq n} \|e_i\|$. Then

$$\begin{aligned} |\|x\| - \|y\|| &\leq \|x - y\| = \left\| \sum_{i=1}^n (x_i - y_i) e_i \right\| \\ &\leq C \sum_{i=1}^n |x_i - y_i| \leq C \sqrt{n} \|x - y\|_2. \end{aligned}$$

Cauchy-Schwarz

Let $\bar{B} \subset \mathbb{R}^n$ be closed unit ball in the Euclidean norm and $S \subset \bar{B}$ its boundary ($S = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$). Then $\|\cdot\|$ is a strictly positive continuous function on S . Hence,