

① since S is compact, it attains its minimal and maximal values $C_1 > 0$ and $C_2 > 0$, resp. Therefore $\|x\| \geq C_1$ and $\|x\| \leq C_2$ if $\|x\|_2 = 1$, that is,
 $\|x\| \geq C_1 \|x\|_2$ and $\|x\| \leq C_2 \|x\|_2 \quad \forall x \in \mathbb{R}^n$. □

Corollary

Assume X and Y are normed spaces, X is f.d., $T: X \rightarrow Y$ is a linear operator. Then T is bounded.

Proof

Define a new norm on X by $\|x\|' \stackrel{\text{def}}{=} \|x\| + \|Tx\|$.
 Then $\|Tx\| \leq \|x\|' \quad \forall x \in X$. The new norm is equivalent to the original one, so
 $\|x\|' \leq C \|x\| \quad \forall x \in X$ for some $C > 0$.
 Hence $\|Tx\| \leq \|x\|' \leq C \|x\|$. □

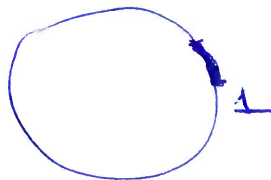
Open mapping theorem (Banach-Schauder)

Assume X, Y are Banach spaces, $T: X \rightarrow Y$ is a surjective bounded linear map. Then T is open.

Recall that a map between metric (or topological) spaces is called open if the image of every open set is open.

Example

Consider $f: [0, 1] \rightarrow \mathbb{T} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| = 1\}$,
 $f(t) = e^{2\pi i t}$. Then f is continuous,
bijective, but not open.



We need some preparation.

Consider a metric space (X, d) .

A subset $A \subset X$ is called nowhere dense if $\bar{A}$ has empty interior. A subset $A \subset X$

is said to be of first category, or meager, if it is the union of countably many nowhere dense sets. A subset $A \subset X$ is called a set of second category if it is not meager. A subset $A \subset X$ is called residual, or comeager, if its complement is meager.

Baire category theorem

Assume (X, d) is a complete metric space. Then X is not a meager subset of itself (so it is a subset of second category).

Proof

Assume X is meager, so $X = \bigcup_{n=1}^{\infty} A_n$,
 $A_n \subset X$ is nowhere dense $\forall n$.

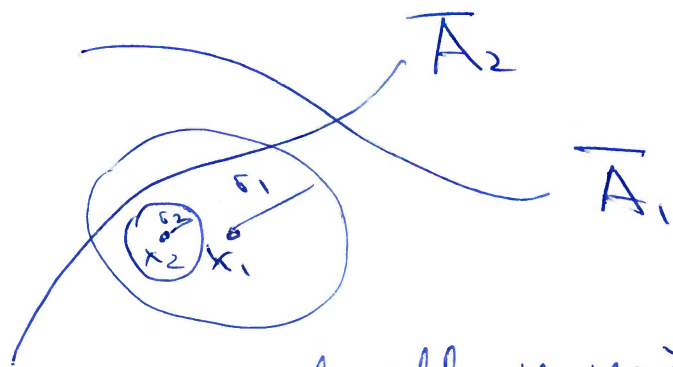
We will construct by induction a sequence of points $x_n \in X$ and a sequence of numbers $\delta_n > 0$ s.t.

$$B_{r_1}(x_1) \supset B_{r_2}(x_2) \supset \dots, \overline{B_{r_n}(x_n)} \cap A_n = \emptyset, r_n \leq \frac{1}{2^n}.$$

As $\overline{A_1}$ has empty interior, there is $x_1 \in X \setminus \overline{A_1}$. Then choose $0 < \delta_1 \leq \frac{1}{2}$ s.t.

$B_{2\delta_1}(x_1) \cap \overline{A_1} = \emptyset$, As $\overline{B_{\delta_1}(x_1)} \subset B_{2\delta_1}(x_1)$, we have $\overline{B_{\delta_1}(x_1)} \cap A_1 = \emptyset$.

As $\overline{A_2}$ has empty interior, we cannot have $B_{\delta_1}(x_1) \subset \overline{A_2}$. Take any $x_2 \in B_{\delta_1}(x_1) \setminus \overline{A_2}$. Choose $\delta_2 \leq \frac{1}{4}$ s.t. $B_{\delta_2}(x_2) \subset B_{\delta_1}(x_1)$ and $B_{2\delta_2}(x_2) \cap \overline{A_2} = \emptyset$. And so on.



For any n_0 and all $n, m \geq n_0$ we have

$x_n, x_m \in B_{r_{n_0}}(x_{n_0})$, so

$$d(x_n, x_m) < 2r_{n_0} \leq \frac{1}{2^{n_0-1}}.$$

Therefore $(x_n)_n$ is a Cauchy sequence.

As X is complete, $x_n \rightarrow x$ for some $x \in X$.

Then $x \in \bigcap_{n=1}^{\infty} \overline{B_{r_n}(x_n)}$, hence $x \notin A_n \forall n$, contradicting that $X = \bigcup_{n=1}^{\infty} A_n$. $\square$

Corollary

Any residual subset $A \subset X$ is dense in X .

Proof

We want to show that for any $x \in X$ and $\delta > 0$ we have $A \cap B_\delta(x) \neq \emptyset$.

By assumption, the set $B = X \setminus A$ is meagre, so $B = \bigcup_{n=1}^{\infty} B_n$, $B_n \subset X$ is nowhere dense.

Consider the complete metric space $Y = \overline{B_\delta(x)}$. Then $B_n \cap Y$ is nowhere dense in Y . By Baire's theorem, $Y \neq \bigcup_{n=1}^{\infty} (B_n \cap Y) = B \cap Y$, so $A \cap Y \neq \emptyset$, that is $A \cap \overline{B_\delta(x)} \neq \emptyset$. Since this is true for all $\delta > 0$ and $\overline{B_\delta(x)} \subset B_{\delta'}(x)$ for $\delta < \delta'$, we conclude that $A \cap B_\delta(x) \neq \emptyset$ for all $\delta > 0$. $\square$

Proof of the open mapping theorem

Assume X, Y are Banach spaces, $T: X \rightarrow Y$ is bounded and surjective.

Consider the open unit ball B in X ,

$$B = B_1(0) = \{x \in X : \|x\| < 1\}.$$

Then $X = \bigcup_{n=1}^{\infty} nB$, hence $Y = \bigcup_{n=1}^{\infty} nT(B)$.

By Baire's theorem, $nT(B) \subset Y$ is not nowhere dense for some $n \geq 1$, hence $tT(B) \subset Y$ is not nowhere dense $\forall t > 0$.

hence there exists $y \in Y$ and $\delta > 0$ s.t. ②

$B_\delta(y) \subset \overline{\frac{1}{2}T(B)}$. Then

$$\begin{aligned} B_\delta(0) &= B_\delta(y) - y \subset \overline{\frac{1}{2}(T(B) - T(b))} \\ &= \overline{\frac{1}{2}T(B-b)} \subset \overline{T(B)}. \end{aligned}$$

It follows that

$$B_t(0) \subset \frac{t}{\delta} \overline{T(B)} \quad \forall t > 0. \quad (*)$$

We claim that in fact

$$B_t(0) \subset \frac{t}{\delta} T(B) \quad \forall t > 0,$$

equivalently, $B_\delta(0) \subset T(B)$. Indeed,

take $y \in B_\delta(0)$. Choose ε , s.t. $\frac{\|y\|}{\delta} < \varepsilon < 1$

and then choose $\varepsilon_n > 0$ for $n \geq 2$ s.t.

$\sum_{n=1}^{\infty} \varepsilon_n < 1$. Using (*) we can construct

by induction elements $x_n \in B_{\varepsilon_n}(0)$ s.t.

$$\|y - \sum_{k=1}^n T x_k\| < \varepsilon_{n+1} \delta$$

(so that $y - \sum_{k=1}^n T x_k \in B_{\varepsilon_{n+1}\delta}(0) \subset \varepsilon_{n+1} \overline{T(B)} = \overline{T(B_{\varepsilon_{n+1}}(0))}$)

Put $x = \sum_{n=1}^{\infty} x_n$, so that $\|x\| \leq \sum_{n=1}^{\infty} \|x_n\| < \sum_{n=1}^{\infty} \varepsilon_n < 1$.

Then $Tx = \lim_{n \rightarrow \infty} \sum_{k=1}^n T x_k = y$, proving our

claim.

Now, if $U \subset X$ is open and $x \in U$,

take $\varepsilon > 0$ s.t. $B_\varepsilon(x) \subset U$. Then $T(U)$ contains

$$T(B_\varepsilon(x)) = Tx + \varepsilon T(B) \supset Tx + B_{\varepsilon\delta}(0) = B_{\varepsilon\delta}(Tx).$$



