

Corollary

Assume X, Y are Banach spaces, $T: X \rightarrow Y$ is a bijective bounded linear map. Then $T^{-1}: Y \rightarrow X$ is bounded.

Proof

By the open mapping theorem we have $T(B_1(0)) \supset B_\delta(0)$ for some $\delta > 0$. That is, $T^{-1}(B_\delta(0)) \subset B_1(0)$, which means that $\|T^{-1}y\| < 1$ whenever $\|y\| < \delta$. ~~Then~~ ^{Then} $\|T^{-1}y\| \leq \frac{1}{\delta} \|y\|$ for all y . $\square$

Def

If X, Y are normed spaces, then we say that a linear map $T: X \rightarrow Y$ is an isomorphism if T is bijective and both T and T^{-1} are bounded.

Therefore for Banach spaces, if $T: X \rightarrow Y$ is bijective and bounded, then it is an isomorphism.

Corollary

Assume X is a vector space and $\|\cdot\|, \|\cdot\|'$ are two norms on X s.t. $(X, \|\cdot\|), (X, \|\cdot\|')$ are Banach spaces and $\|x\|' \leq C\|x\| \forall x \in X$ for some $C > 0$. Then $\|\cdot\|' \sim \|\cdot\|$.

Proof

This follows from the previous corollary applied to $\text{id}: (X, \|\cdot\|) \rightarrow (X, \|\cdot\|')$. $\square$

Given a linear map $T: X \rightarrow Y$, the graph of T is the subspace

$$\Gamma(T) \stackrel{\text{def}}{=} \{(x, Tx) : x \in X\} \subset X \oplus Y$$

If X, Y are normed spaces, we equip $X \oplus Y$ with the norm $\|(x, y)\| = \|x\| + \|y\|$.

Remark

If X, Y are Hilbert spaces, we instead consider the norm $\|(x, y)\| = (\|x\|^2 + \|y\|^2)^{1/2}$. These norms are anyway equivalent (exercise).

Corollary ("closed graph theorem")

Assume X, Y are Banach spaces, $T: X \rightarrow Y$ is a linear map. Then T is bounded if and only if $\Gamma(T) \subset X \oplus Y$ is a closed subspace (that is, if $x_n \rightarrow x$, $Tx_n \rightarrow y$, then $y = Tx$).

Proof

It is clear that if T is bounded, then $\Gamma(T)$ is closed. Conversely, assume $\Gamma(T)$ is closed. Since X, Y are Banach spaces, $X \oplus Y$ and hence $\Gamma(T)$ are Banach spaces as well. Consider the linear map $p: \Gamma(T) \rightarrow X$, $p(x, Tx) = x$. It is bounded and bijective. Hence p^{-1} is bounded, that is, $\|x\| + \|Tx\| \leq C\|x\|$ $\forall x$ for some $C > 0$. Then $\|Tx\| \leq (C-1)\|x\|$. □

Assume X is a normed space and Y is a closed subspace. Then X/Y can be equipped with the norm

$$\|x+Y\| \stackrel{\text{def}}{=} \text{dist}(x, Y) = \inf_{y \in Y} \|x-y\| = \inf_{y \in Y} \|x+y\|$$

Note that $\|x+Y\| > 0$ for $x \notin Y$ by closedness of Y in X .

Proposition

If X is a Banach space and $Y \subset X$ is a closed subspace, then X/Y is a Banach space.

Proof

Assume $(x_n+Y)_{n=1}^{\infty}$ is a Cauchy sequence in X/Y . By passing to a subsequence we may assume that $d(x_n+Y, x_{n-1}+Y) < \frac{1}{2^n}$ $\forall n \geq 2$, that is, $\inf_{y \in Y} \|x_n - x_{n-1} - y\| < \frac{1}{2^n}$. By replacing inductively x_n by $x_n + y_n$ for some $y_n \in Y$ we may assume that $\|x_{n+1} - x_n\| < \frac{1}{2^n}$. Then $(x_n)_n$ is a Cauchy sequence in X . Hence $x_n \rightarrow x$ for some $x \in X$. Then $x_n+Y \rightarrow x+Y$ in X/Y . $\square$

Applying the open mapping theorem we get the following result.

Corollary

Assume X, Y are Banach spaces, $T: X \rightarrow Y$ is a surjective bounded linear map. Then $\ker T \subset X$ is closed and we have an isomorphism $X/\ker T \xrightarrow{\sim} Y$, $x + \ker T \mapsto Tx$, of Banach spaces.

Proof
That $\ker T$ is closed is immediate from continuity of T . We then have a linear isomorphism $\tilde{T}: X/\ker T \xrightarrow{\sim} Y$, $\tilde{T}(x + \ker T) = Tx$. $\tilde{T}$ is bounded, since for every $x' \in \ker T$ we have

$$\|Tx\| = \|T(x+x')\| \leq \|T\| \cdot \|x+x'\|,$$

hence, taking the infimum over $x' \in \ker T$, we get $\|Tx\| \leq \|T\| \cdot \|x + \ker T\|$, so $\|\tilde{T}\| \leq \|T\|$. Since $X/\ker T$ and Y are Banach spaces, it follows that $\tilde{T}^{-1}$ is bounded. $\square$

Another application of the Baire category theorem gives the following result, the third of the fundamental principles of functional analysis.

Uniform boundedness principle (Banach-Steinhaus) ⁽³⁾

Assume X is a Banach space and Y is a normed space, $(T_i)_{i \in I}$ is a collection of bounded linear operators $T_i : X \rightarrow Y$ s.t.
 $\sup_i \|T_i x\| < \infty \quad \forall x \in X$. Then $\sup_i \|T_i\| < \infty$.

Proof

For $n \geq 1$, consider $A_n \stackrel{\text{def}}{=} \{x : \|T_i x\| \leq n \quad \forall i\}$.
Then each set A_n is closed and, by assumption, $\bigcup_{n=1}^{\infty} A_n = X$. Since X is complete, by Baire category theorem, there is $n \geq 1$ s.t. $B_\delta(x) \subset A_n$ for some $x \in X$ and $\delta > 0$.
Then $B_\delta(0) = B_\delta(x) - x \subset A_{2n}$, that is,
 $\|T_i\| \leq \frac{2n}{\delta} \quad \forall i.$ □

Corollary

Assume X is a Banach space, Y is a normed space, and $(T_n)_{n=1}^{\infty}$ is a sequence of bounded linear operators s.t.

$Tx \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} T_n x$ exists $\forall x \in X$. Then

$T : X \rightarrow Y$ is again a bounded linear operator.

Warning: $\|T - T_n\| \not\rightarrow 0$ in general.

Proof

Linearity of T is immediate.

Since $\lim_{n \rightarrow \infty} T_n x$ exists $\forall x$, we have
 $\sup_n \|T_n x\| < \infty \quad \forall x$. Hence $C \stackrel{\text{def}}{=} \sup_n \|T_n\|$
is finite by the uniform boundedness
principle. Then $\|Tx\| \leq C\|x\| \quad \forall x$. $\square$