

As an application consider Fourier series of functions on $(0,1)$. Define $e_n(x) = e^{2\pi i n x}$,
 $c_n(f) = \int_0^1 f(x) \overline{e_n(x)} dx$ (for $f \in L^1(0,1)$).

It is known that

$$\sum_{n=-N}^N c_n(f) e_n(x) \xrightarrow{N \rightarrow \infty} f(x)$$

- in $L^2(0,1)$, if $f \in L^2(0,1)$;
- for a.e. $x \in (0,1)$, if $f \in L^2(0,1)$ (Carleson);
- for every $x \in (0,1)$ s.t. $f'(x)$ exists, if $f \in C[0,1]$.

Proposition

Consider the space $X = \{f \in C[0,1] : f(0) = f(1)\}$ (with the supremum norm). Fix $x_0 \in [0,1]$.
 Then the set of $f \in X$ s.t. $\left(\sum_{n=-N}^N c_n(f) e_n(x_0)\right)_{N=1}^\infty$ is bounded is meager in X .

Proof

By translating by $-x_0$ we may assume that $x_0 = 0$. We will only show that there exists one f s.t. $\left(\sum_{n=-N}^N c_n(f)\right)_{N=1}^\infty$ is unbounded, leaving the full proof as an exercise.

Assume there is no such $f \in X$.

Define linear functionals, $\varphi_N : X \rightarrow \mathbb{C}$

$$\text{by } \varphi_N(f) = \sum_{n=-N}^N c_n(f) = \int_0^1 f(x) \sum_{n=-N}^N e_n(x) dx.$$

By assumption, $\sup_N |\varphi_N(f)| < \infty \quad \forall f \in X$.

By the uniform boundedness principle it follows that $\sup_n \|\varphi_n\| < \infty$. On the other hand,

$$\begin{aligned}\|\varphi_n\| &= \int_0^1 \left| \sum_{k=-n}^n c_k(x) \right| dx = \int_0^1 \left| \frac{e^{2i\bar{u}(2n+1)x} - 1}{e^{2i\bar{u}x} - 1} \right| dx \\ &= \int_0^1 \left| \frac{\sin 2\bar{u}(n+\frac{1}{2})x}{\sin 2\bar{u}\frac{x}{2}} \right| dx \\ &\geq \frac{1}{2\bar{u}} \int_0^1 \frac{|\sin 2\bar{u}(n+\frac{1}{2})x|}{x} dx \\ &= \frac{1}{2\bar{u}} \int_0^{2\bar{u}(n+\frac{1}{2})} \frac{|\sin x|}{x} dx \xrightarrow{n \rightarrow \infty} +\infty,\end{aligned}$$

Since $|\sin x| \geq \varepsilon > 0$ for $x \in (0, +\infty) \setminus \bigcup_n (2\bar{u}n - \delta, 2\bar{u}n + \delta)$

and $\int_{(0, +\infty) \setminus \bigcup_n (2\bar{u}n - \delta, 2\bar{u}n + \delta)} \frac{dx}{x} = +\infty$ for any $\delta \in (0, \frac{\pi}{2})$ and suitable $\varepsilon > 0$. □

The following result is another useful application of the uniform boundedness principle.

Proposition

Assume X is a Banach space, Y and Z are normed spaces, and $B: X \times Y \rightarrow Z$ is a bilinear map that is continuous in each variable. Then there exists $C \geq 0$ s.t., $\|B(x, y)\| \leq C \|x\| \cdot \|y\| \forall x \in X, y \in Y$.

In particular, B is jointly continuous.

The bilinear maps satisfying the above inequality are called bounded.

Proof

For every $x \in X$, the operator $B(x, \cdot): Y \rightarrow Z$ is bounded, so $\sup_{\|y\| \leq 1} \|B(x, y)\| < \infty$. Therefore

the collection of operators $(B(\cdot, y): X \rightarrow Z)_{\|y\| \leq 1}$ is pointwise bounded. Hence it is uniformly bounded, that is, there is $C \geq 0$ s.t.
$$\|B(x, y)\| \leq C\|x\| \quad \forall x \in X \quad \forall y \in Y \text{ with } \|y\| \leq 1.$$

Then $\|B(x, y)\| \leq C\|x\| \cdot \|y\| \quad \forall x \in X \quad \forall y \in Y$.

The map $B: X \times Y \rightarrow Z$ is continuous, since

$$\begin{aligned} \|B(x, y) - B(x', y')\| &\leq \|B(x - x', y)\| + \|B(x', y - y')\| \\ &\leq C\|x - x'\| \cdot \|y\| + C\|x'\| \cdot \|y - y'\|. \end{aligned}$$

Bounded bilinear forms ($Z = \mathbb{C}$ or $\mathbb{R}$) are closely related to bounded linear operators. For example:

Proposition

Given complex normed spaces X, Y , there is a 1-to-1 correspondence between bounded bilinear forms $B: X \times Y \rightarrow \mathbb{C}$ and bounded linear operators $A: X \rightarrow Y^*$. Namely, every $A \in L(X, Y^*)$ defines

$$B(x, y) = (Ax)(y).$$

Proof

The inverse map is $B \mapsto (x \mapsto B(x, \cdot))$.

Recall that for Hilbert spaces we have an antilinear isomorphism $H \cong H^*$, $\xi \mapsto (\cdot, \xi)$ (equivalently, an isomorphism $H \subset H^*$). A ~~linear~~ map $B: H \times K \rightarrow \mathbb{C}$ is called sesquilinear if it is linear in the first variable and antilinear in the second variable. Similarly to the above proposition we get a 1-1 correspondence between bounded sesquilinear forms $H \times K \rightarrow \mathbb{C}$ and bounded operators $H \rightarrow K$: given $A: H \rightarrow K$, we define $B(x, y) = (Ax, y)$.

Given $A \in L(H, K)$, we can consider the sesquilinear form $B: K \times H \rightarrow \mathbb{C}$, $B(y, x) = (y, Ax) = \overline{(Ax, y)}$. It corresponds to a bounded linear operator $A^*: K \rightarrow H$, called the adjoint of A . Thus,

$$(Ax, y) = (x, A^*y) \text{ for } x \in H, y \in K.$$

Proposition

The map $A \mapsto A^*$ is antilinear, antimultiplicative ($(AB)^* = B^*A^*$ for $A \in L(H, K), B \in L(M, H)$), involutive ($A^{**} = A$), and $\|A^*\| = \|A\|$.

Proof Let us only check the last property:

$$\|A\| = \sup_{\|x\| \leq 1} \|Ax\| = \sup_{\substack{\|x\| \leq 1 \\ \|y\| \leq 1}} |(Ax, y)| = \sup_{\substack{\|x\| \leq 1 \\ \|y\| \leq 1}} |(A^*y, x)| = \|A^*\|.$$



(3)

For general normed spaces, a close relative of the adjoint operator is the transpose $T^t: Y^* \rightarrow X^*$ of $T: X \rightarrow Y$ (T bounded), defined by

$$(T^t f)(x) = f(Tx) \text{ for } x \in X, f \in Y^*.$$

Remark

T^t is often denoted by T^* , but we won't use this notation in order to distinguish between adjoint and transpose operators.

Proposition

For normed spaces X, Y , the map $L(X, Y) \rightarrow L(Y^*, X^*), T \mapsto T^t$, is linear, antimultiplicative, and $\|T^t\| = \|T\|$. We also have $T^{tt}|_X = T$.

Proof

Let us check the last property.

Given $f \in Y^*$ and $x \in X$, we have

$$|(T^t f)(x)| = |f(Tx)| \leq \|f\| \cdot \|Tx\| \leq \|T\| \cdot \|f\| \cdot \|x\|,$$

so $\|T^t f\| \leq \|T\| \cdot \|f\|$ and therefore $\|T^t\| \leq \|T\|$.

Then $\|T^{tt}\| \leq \|T^t\| \leq \|T\|$, and since

$T^{tt}|_X = T$ we conclude that $\|T^t\| = \|T\|$. $\square$

