

Spectrum of an operator

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Assume X is a complex Banach space.

Def

The spectrum of $T \in L(X)$ is the set

$$\sigma(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible} \}.$$

The point spectrum of $T \in L(X)$ is the set

$$\sigma_p(T) = \{ \lambda \in \mathbb{C} : \ker(T - \lambda I) \neq \{0\} \}.$$

Clearly, $\sigma_p(T) \subset \sigma(T)$.

Example

Consider $H = L^2(0,1)$ and the operator $A \in L(H)$,
 $(Af)(x) = x f(x)$. Then $\sigma(A) = [0,1]$, $\sigma_p(A) = \emptyset$.

For a Banach space X (real or complex),
denote by $GL(X)$ the set of bounded
invertible operators $T: X \rightarrow X$. We know
that if $T \in GL(X)$, then T^{-1} is bounded,
hence $GL(X)$ is a group.

Proposition

Assume X is a Banach space. Then
(1) $GL(X)$ is open in $L(X)$; more precisely,
if $T \in GL(X)$ and $\|S - T\| < \frac{1}{\|T^{-1}\|}$, then
 $S \in GL(X)$;
(2) the map $GL(X) \rightarrow GL(X)$, $T \mapsto T^{-1}$, is
continuous.

Proof

(1) Since $\|ST\| \leq \|S\| \cdot \|T\|$ for any bounded linear operators S and T , it follows that if $\|S\| < 1$, then $\|S^n\| \leq \|S\|^n$ and therefore $\sum_{n=0}^{\infty} S^n$ is an absolutely convergent series in the Banach space $L(X)$. Moreover,

$$(1-S) \sum_{n=0}^N S^n = 1 - S^{N+1} \xrightarrow{N \rightarrow \infty} 1 \text{ and}$$

$$\left(\sum_{n=0}^N S^n \right) (1-S) = 1 - S^{N+1} \xrightarrow{N \rightarrow \infty} 1. \text{ Therefore}$$

$$(1-S)^{-1} = \sum_{n=0}^{\infty} S^n \quad (\text{"Neumann series"})$$

Now, if $T \in GL(X)$ and $\|S-T\| < \frac{1}{\|T^{-1}\|}$,

then $\|T^{-1}(T-S)\| \leq \|T^{-1}\| \|T-S\| < 1$ and then

$$\begin{aligned} S^{-1} &= (T - (T-S))^{-1} = (T(1 - T^{-1}(T-S)))^{-1} \\ &= (1 - T^{-1}(T-S))^{-1} T^{-1}. \end{aligned}$$

(2) let us check first that the map

$T \mapsto T^{-1}$ is continuous at 1 .

Assume $T_n \rightarrow 1$. Write $T_n = 1 - S_n$, so $S_n \rightarrow 0$.

Then $T_n^{-1} = (1 - S_n)^{-1} = \sum_{k=0}^{\infty} S_n^k$, so

$$\|T_n^{-1} - 1\| = \left\| \sum_{k=1}^{\infty} S_n^k \right\| \leq \sum_{k=1}^{\infty} \|S_n\|^k = \frac{\|S_n\|}{1 - \|S_n\|} \xrightarrow{n \rightarrow \infty} 0.$$

For general $T \in GL(X)$, if $T_n \rightarrow T$,

then $T^{-1}T_n \rightarrow 1$, hence $T_n^{-1}T = (T^{-1}T_n)^{-1} \rightarrow 1$,

and then $T_n^{-1} \rightarrow T^{-1}$. □

Corollary

If X is a complex Banach space and $T \in L(X)$, then $\mathcal{B}(T)$ is a closed subset of $\{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$.

Proof

Since $GL(X) \subset L(X)$ is open, the set $\{\lambda \in \mathbb{C} : T - \lambda I \in GL(X)\}$ is open, hence its complement $\mathcal{B}(T)$ is closed.

By the proposition, $(T - \lambda I) \in GL(X)$

whenever $\|T\| < \frac{1}{\|X^{-1}I\|} = |\lambda|$. □

As we will see soon, $\mathcal{B}(T)$ is never empty. The number

$$\begin{aligned} \rho(T) &\stackrel{\text{def}}{=} \sup\{|\lambda| : \lambda \in \mathcal{B}(T)\} \\ &= \max\{|\lambda| : \lambda \in \mathcal{B}(T)\} \end{aligned}$$

is called the spectral radius of T .

Thus, $\rho(T) \leq \|T\|$.

The following criterion of invertibility is often useful.

Lemma

Assume X is a Banach space, $T \in L(X)$. Then T is invertible if and only if the following conditions are satisfied:

- (i) there is $C > 0$ s.t. $\|Tx\| \geq C\|x\| \quad \forall x \in X$;
- (ii) $\text{Im } T \stackrel{\text{def}}{=} \{Tx : x \in X\}$ is dense in X .

Proof

Assume T is invertible. Then $\text{Im } T = X$ and letting $y = Tx$ in the inequality $\|T^{-1}y\| \leq \|T^{-1}\| \|y\|$, we get

$$\|Tx\| \geq \frac{1}{\|T^{-1}\|} \|x\|.$$

Conversely assume (i) and (ii) are satisfied. (i) already implies that $\ker T = 0$. It remains to show that $\text{Im } T = X$, and since $\text{Im } T$ is dense in X by (ii), it suffices to show that $\text{Im } T$ is closed.

Assume $Tx_n \rightarrow y$. Then $(Tx_n)_{n=1}^{\infty}$ is a Cauchy sequence. Since

$$\|x_n - x_m\| \leq \frac{1}{c} \|Tx_n - Tx_m\|,$$

$(x_n)_{n=1}^{\infty}$ is also a Cauchy sequence. Hence $x_n \rightarrow x$ for some x , and then

$$Tx = \lim_{n \rightarrow \infty} Tx_n = y, \text{ so } y \in \text{Im } T. \quad \square$$

Using this let us prove

Proposition

- (1) If H is a Hilbert space and $A \in L(H)$, then $\mathcal{B}(A^*) = \overline{\mathcal{B}(A)} \stackrel{\text{def}}{=} \{\bar{z} : z \in \mathcal{B}(A)\}$.
- (2) If X is a complex Banach space and $T \in L(X)$, then $\mathcal{B}(T^*) = \mathcal{B}(T)$.

Proof

- (1) If $\lambda \notin \mathcal{B}(A)$, so that $S = (A - \lambda I)^{-1}$ exists, then taking adjoints of $S(A - \lambda I) = (A - \lambda I)S = I$,

we get $(A^* - \bar{\lambda}I)S^* = S^*(A^* - \bar{\lambda}I) = I$, so (3)

$\bar{\lambda} \notin b(A^*)$. Therefore $b(A^*) \subset \overline{b(A)}$.

Since $A^{**} = A$, we then also have

$b(A) = b(A^{**}) \subset \overline{b(A^*)}$. Hence $b(A^*) = \overline{b(A)}$.

(2) Similarly to (1), we have

$$b(T) \supset b(T^+) \supset b(T^{++}).$$

Take $\lambda \in b(T)$. By the previous lemma, either there is no $c > 0$ s.t. $\|(T - \lambda I)x\| \geq c\|x\|$ $\forall x \in X$ or $\text{Im}(T - \lambda I)$ is not dense in X .

In the first case, since $T = T^{++}|_X$, there is no $c > 0$ s.t. $\|(T^{++} - \lambda I)x\| \geq c\|x\| \forall x \in X^{**}$,

hence $\lambda \in b(T^+)$. In the second

case $\frac{X}{\overline{\text{Im}(T - \lambda I)}} \neq 0$, hence there exists

$f \in X^*$, $f \neq 0$, but $f = 0$ on $\overline{\text{Im}(T - \lambda I)}$.

Then $f \circ (T - \lambda I) = 0$, that is, $T^+ f = \lambda f$ and therefore $\lambda \in b(T^+)$. $\square$

Given a polynomial $p \in \mathbb{C}[z]$, $p(z) = a_n z^n + \dots + a_1 z + a_0$, we define

$$p(T) = a_n T^n + \dots + a_1 T + a_0 I.$$

We then have $(p+q)(T) = p(T) + q(T)$,

$$(pq)(T) = p(T)q(T).$$

Theorem ("Spectral mapping theorem for polynomial functions")

Assume X is a complex Banach space,
 $T \in L(X)$ and $p \in \mathbb{C}[z]$. Then

$$\sigma(p(T)) = p(\sigma(T)) (= \{p(z) : z \in \sigma(T)\}).$$

Proof

If $\deg p = 0$, the result is immediate.

Assume $\deg p = n \geq 1$. Take $\lambda \in \mathbb{C}$ and
write $p(z) - \lambda = a_n(z - \lambda_1) \dots (z - \lambda_n)$. Then
 $p(T) - \lambda I = a_n(T - \lambda_1 I) \dots (T - \lambda_n I)$. Since
the operators $T - \lambda_i I$ pairwise commute, we
have that $p(T) - \lambda I$ is invertible

$(\Rightarrow) T - \lambda_i I$ is invertible for $i = 1, \dots, n$

$(\Rightarrow) \lambda_i \notin \sigma(T)$ for $i = 1, \dots, n$

$(\Rightarrow)$ no z s.t. $p(z) = \lambda$ lies in $\sigma(T)$

$(\Rightarrow) \lambda \notin p(\sigma(T))$. □