

Given a complex Banach space X and $T \in L(X)$, the resolvent set of T is ①

$$\rho(T) \stackrel{\text{def}}{=} \mathbb{C} \setminus \sigma(T).$$

We already know that this is an open set containing $\{z: |z| > \|T\|\}$. The resolvent of T is the map

$$R_T: \rho(T) \rightarrow L(X), R_T(z) = (T - zI)^{-1}.$$

We already know that this map is continuous. Much more is true.

Def

Given an open set $\Omega \subset \mathbb{C}$ and a complex Banach space Y , a vector-valued function $f: \Omega \rightarrow Y$ is called analytic if

$$f'(z_0) \stackrel{\text{def}}{=} \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \in Y \text{ exists } \forall z_0 \in \Omega.$$

Lemma

If X is a complex Banach space and $T \in L(X)$, then $R_T: \rho(T) \rightarrow L(X)$ is analytic.

Proof

For $z, z_0 \in \rho(T)$, we have

$$\begin{aligned} R_T(z) - R_T(z_0) &= (T - z_0 I)^{-1} - (T - z I)^{-1} \\ &= (T - z_0 I)^{-1} ((T - z_0 I) - (T - z I)) (T - z I)^{-1} \\ &= (z - z_0) R_T(z_0) R_T(z) \end{aligned}$$

This is called the "resolvent identity".

It follows that $R_T'(z_0) = R_T(z_0)^2$.

We also observe the following.

Lemma

$R_T(z) \rightarrow 0$ as $|z| \rightarrow \infty$.

Proof

For $|z| > \|T\|$ we have

$$(T - zI)^{-1} = -\frac{1}{z} \left(1 - \frac{T}{z}\right)^{-1} = -\frac{1}{z} \sum_{n=0}^{\infty} \frac{T^n}{z^n}, \text{ so}$$

$$\|R_T(z)\| \leq \frac{1}{|z|} \cdot \frac{1}{1 - \frac{\|T\|}{|z|}} = \frac{1}{|z| - \|T\|} \xrightarrow{|z| \rightarrow \infty} 0.$$

Theorem

For any complex Banach space $X \neq 0$ and $T \in L(X)$, the spectrum $\sigma(T)$ is a nonempty bounded closed subset of $\mathbb{C}$, and $\sigma(T) = \lim_{n \rightarrow \infty} \|T^n\|^{1/n}$.

Proof

Observe that for any $f \in L(X)^*$, the function $f \circ R_T : \rho(T) \rightarrow \mathbb{C}$ is analytic and $(f \circ R_T)'(z) = f(R_T'(z))$.

Assume $\sigma(T) = \emptyset$. Then, for any $f \in L(X)^*$, the function $f \circ R_T$ is analytic on $\mathbb{C}$ and vanishes at ∞ . By Liouville's theorem it follows that it is zero, so $f(R_T(z)) = 0 \quad \forall f \in L(X)^* \quad \forall z \in \mathbb{C}$. Hence $R_T(z) = 0 \quad \forall z \in \mathbb{C}$, which is nonsense. Thus, $\sigma(T) \neq \emptyset$.

we already know that $\sigma(T)$ is closed and $r(T) \leq \|T\|$. It remains to show that $\sigma(T) = \lim_{n \rightarrow \infty} \|T^n\|^{1/n}$. ②

By the spectral mapping theorem, we have $\sigma(T^n) = \{z^n : z \in \sigma(T)\}$. Hence $r(T^n) = r(T)^n$ and then $\sigma(T) = \sigma(T^n)^{1/n} \leq \|T^n\|^{1/n}$, so that $\sigma(T) \subseteq \liminf_{n \rightarrow \infty} \|T^n\|^{1/n}$.

Consider the function $\varphi(z) = \frac{1}{z} R_T(\frac{1}{z})$. It is well-defined and analytic on $\{z : 0 < |z| < \frac{1}{r(T)}\}$. For $|z| < \frac{1}{\|T\|}$ we have

$$\varphi(z) = \frac{1}{z} (I - zT)^{-1} = -(I - zT)^{-1} = -\sum_{n=0}^{\infty} z^n T^n,$$

so letting $\varphi(0) = -I$ we get an analytic function on $\{z : |z| < \frac{1}{r(T)}\}$.

For any $f \in L(X)^*$, the function $f \circ \varphi$ is also analytic on $\{z : |z| < \frac{1}{r(T)}\}$ and $(f \circ \varphi)(z) = -\sum_{n=0}^{\infty} z^n f(T^n)$ for $|z| < \frac{1}{\|T\|}$.

Fix $\varepsilon > 0$ and consider the circle $\Gamma_\varepsilon = \{z : |z| = r(T) + \varepsilon\}$ with counterclockwise orientation. Let $M_\varepsilon = \sup_{z \in \Gamma_\varepsilon} \|\varphi(z)\|$.

Then

$$f(T^n) = -\frac{1}{2\pi i} \oint_{\Gamma_\varepsilon} \frac{(f \circ \varphi)(z)}{z^{n+1}} dz.$$

$$\text{Hence } |f(T^n)| \leq \frac{\|f\| \cdot M_\varepsilon}{\left(\frac{1}{r(T) + \varepsilon}\right)^n} = \|f\| \cdot M_\varepsilon (r(T) + \varepsilon)^n.$$

Since this is true for all $f \in L(X)^*$, it follows that

$$\|T^n\| \leq M_\varepsilon (\sigma(T) + \varepsilon)^n,$$

that is,

$$\|T^n\|^{1/n} \leq M_\varepsilon^{1/n} (\sigma(T) + \varepsilon),$$

and hence

$$\limsup_{n \rightarrow \infty} \|T^n\|^{1/n} \leq \sigma(T) + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we thus get

$$\limsup_{n \rightarrow \infty} \|T^n\|^{1/n} \leq \sigma(T), \text{ hence}$$

$$\sigma(T) = \lim_{n \rightarrow \infty} \|T^n\|^{1/n}.$$

Proposition

If H is a Hilbert space and $A \in L(H)$,

then $\|A^*A\| = \|A\|^2$ (C^* -identity)

Proof

$$\|A^*A\| \leq \|A^*\| \|A\| = \|A\| \cdot \|A\| = \|A\|^2.$$

On the other hand,

$$\begin{aligned} \|A\|^2 &= \sup_{\|x\| \leq 1} \|Ax\|^2 = \sup_{\|x\| \leq 1} (Ax, Ax) \\ &= \sup_{\|x\| \leq 1} (A^*Ax, x) \leq \sup_{\|x\| \leq 1} \|A^*Ax\| \cdot \|x\| \\ &\quad \text{Cauchy-Schwarz} \end{aligned}$$

$$\|A^*A\| = \|A\|^2.$$

□

Def

An operator $A \in L(H)$ for a Hilbert space H is called normal if $A^*A = AA^*$. It is called self-adjoint if $A^* = A$.

Corollary

③

If H is a Hilbert space and $A \in L(H)$ is normal, then $\sigma(A) = \|A\|$.

Proof

If $A = A^*$, then A^n is also self-adjoint, hence $\|A^n\| = \|(A^n)^* A^n\| = \|A^n\|^2$. By using this repeatedly we get $\|A^{2^n}\| = \|A\|^{2^n} \forall n$.

Assume now that A is normal. Then

A^*A is self-adjoint. Hence

$$\begin{aligned}\|A^{2^n}\| &= \|(A^{2^n})^* A^{2^n}\|^{1/2} = \|(A^*A)^{2^n}\|^{1/2} \\ &= \|A^*A\|^{2^{n-1}} = \|A\|^{2^n}.\end{aligned}$$

$$\text{Then } \sigma(A) = \lim_{n \rightarrow \infty} \|A^{2^n}\|^{1/2^n} = \|A\|. \quad \square$$

Remark

As a byproduct we get that if A is normal, then $\|A^n\| = \|A\|^n \forall n$. Indeed, $\|A^n\| = \sigma(A^n) = \sigma(A)^n = \|A\|^n$.

