

Compact and Fredholm operators

①

Def

A metric space K is called precompact if its completion is compact.

It is known that the following conditions are equivalent:

- K is precompact;
- K is totally bounded (that is, $\forall \varepsilon > 0$ K can be covered by finitely many balls of radius ε);
- any sequence in K has a Cauchy subsequence.

Theorem

Assume X is a normed space. Then the unit ball in X is precompact if and only if X is f.d.

For the proof we need

Lemma (Riesz)

Assume X is a normed space, $Y \subset X$ is a closed proper (that is, $Y \neq X$) subspace, $0 < \delta < 1$. Then there exists $x \in X$ s.t. $\|x\| = 1$ and $d(x, Y) > \delta$.

Proof

Take any $x' \in X \setminus Y$. Let $r \stackrel{\text{def}}{=} d(x', Y) > 0$. For $\varepsilon > 0$, ~~xxxx~~ to be specified later, choose $y \in Y$ s.t. $\|x' - y\| < r + \varepsilon$, and let $x = \frac{x' - y}{\|x' - y\|}$.

Then $\|x\| \geq 1$ and

$$\begin{aligned} d(x, Y) &= \frac{1}{\|x' - y\|} d(x' - y, Y) = \frac{1}{\|x' - y\|} d(x', Y) \\ &= \frac{r}{\|x' - y\|} > \frac{r}{r + \varepsilon}. \end{aligned}$$

We see that if ε is small enough so that $\frac{r}{r + \varepsilon} > \delta$, then x has the required properties. $\square$

Proof of the theorem

If X is f.d., then $X \cong \mathbb{R}^n$ as a real normed vector space. Since every bounded subset of $\mathbb{R}^n$ is precompact, the unit ball in X is precompact.

Assume now that X is not f.d. Fix $\delta \in (0, 1)$. We will construct by induction elements $x_n \in X$ s.t. $\|x_n\| \geq 1$ and $\|x_n - x_m\| > \delta$ for all $n \neq m$. As x_1 take any unit vector. Assume $x_1, \dots, x_n$ are constructed. Consider

$Y_n = \text{span}\{x_1, \dots, x_n\}$. The space Y_n is f.d., hence complete, hence it is closed in X . By Riesz's lemma we can find x_{n+1} s.t. $\|x_{n+1}\| \geq 1$ and $d(x_{n+1}, Y_n) > \delta$. In particular, $\|x_{n+1} - x_m\| > \delta$ for $m = 1, \dots, n$.

Obviously, the sequence $(x_n)_{n=1}^{\infty}$ does not have a Cauchy subsequence, so the unit ball in X is not precompact. $\square$

Def

For normed spaces X and Y , an operator $T \in L(X, Y)$ is called compact if the image of the unit ball under T is precompact.

(Equivalently, for any bounded sequence $(x_n)_{n=1}^{\infty}$ in X , the sequence $(Tx_n)_{n=1}^{\infty}$ has a Cauchy subsequence.)

We denote by $K(X, Y) \subset L(X, Y)$ the subset of compact operators. We write $K(X)$ for $K(X, X)$.

Proposition

- (1) $K(X, Y) \subset L(X, Y)$ is a closed subspace.
- (2) If $S \in L(Y, Z)$ and $T \in K(X, Y)$, then $ST \in K(X, Z)$. Similarly, if $S \in L(Z, X)$ and $T \in K(X, Y)$, then $TS \in K(Z, Y)$.

Proof

- (1) It is immediate that $K(X, Y)$ is closed under multiplication by scalars.

To see that it is closed under addition, take $S, T \in K(X, Y)$. Assume $(x_n)_{n=1}^{\infty}$

is a bounded sequence in X . Then

$(Tx_n)_{n=1}^{\infty}$ has a Cauchy subsequence

$(Tx_{n_k})_{k=1}^{\infty}$. Then $(Sx_{n_k})_{k=1}^{\infty}$ has a Cauchy

subsequence $(Sx_{n_{k_\ell}})_{\ell=1}^{\infty}$. Then $(Sx_{n_{k_\ell}} + Tx_{n_{k_\ell}})_{\ell=1}^{\infty}$

is a Cauchy subsequence of $(Sx_n + Tx_n)_{n=1}^{\infty}$.

~~(1)~~ Assume now that $T_n \in K(X, Y)$ and $T_n \rightarrow T \in L(X, Y)$. Consider the (open) unit ball B in X . We will show that $TB \subset Y$ is totally bounded. Take $\varepsilon > 0$.

Choose n s.t. $\|T - T_n\| < \frac{\varepsilon}{2}$. Since $T_n B$ is totally bounded, there exist balls

$B_{\frac{\varepsilon}{2}}(x_1), \dots, B_{\frac{\varepsilon}{2}}(x_m)$ covering $T_n B$.

Then $TB \subset \bigcup_{k=1}^m B_{\varepsilon}(x_k)$. Indeed, take any $x \in B$. Then $T_n x \in B_{\frac{\varepsilon}{2}}(x_k)$ for some k , that is, $\|T_n x - x_k\| < \frac{\varepsilon}{2}$.

Then $\|Tx - x_k\| \leq \|Tx - T_n x\| + \|T_n x - x_k\|$
 $\leq \|T - T_n\| + \|T_n x - x_k\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Therefore TB is indeed totally bounded, so T is compact.

(2) Take $T \in K(X, Y)$, $S \in L(Y, Z)$.

If $(x_n)_n$ is bounded, then $(Tx_n)_n$ has a Cauchy subsequence $(Tx_{n_k})_k$.

Then $(STx_{n_k})_k$ is also Cauchy. Hence $ST \in K(X, Z)$.

Similarly, if $T \in K(X, Y)$, $S \in L(Z, X)$ and $(z_n)_n$ is a bounded sequence in Z , then $(Sz_n)_n$ is also bounded, hence $(TSz_n)_n$ has a Cauchy subsequence $(TSz_{n_k})_k$. Thus, $TS \in K(Z, Y)$. $\square$

Def

(8)

A bounded operator $T: X \rightarrow Y$ is said to be of finite rank if $\text{Im } T$ is f.d.

We denote by $F(X, Y) \subset L(X, Y)$ the set of finite rank operators. It is easy to see that it is a subspace. We write $F(X)$ for $F(X, X)$.

Lemma

$$F(X, Y) \subset K(X, Y).$$

Proof

Indeed, any bounded subset of a f.d. normed space is precompact. $\square$

It follows that $\overline{F(X, Y)} \subset K(X, Y)$. As we will see soon, we often have $\overline{F(X, Y)} = K(X, Y)$.

Lemma

A linear operator $T: X \rightarrow Y$ lies in $F(X, Y)$ if and only if T has the form $Tx = \sum_{k=1}^n f_k(x) y_k$ for some $y_1, \dots, y_n \in Y$ and $f_1, \dots, f_n \in X^*$.

Proof

"If" is clear. For "only if", consider $T \in F(X, Y)$. Choose a basis $y_1, \dots, y_n$ in $\text{Im } T$. Let $g_1, \dots, g_n \in (\text{Im } T)^*$ be the dual basis, that is, $g_i(y_j) = \delta_{ij}$. Then $y = \sum_{k=1}^n g_k(y) y_k \quad \forall y \in \text{Im } T$ (since this is true for all y_k).

hence, for all $x \in X$, we have

$$Tx = \sum_{k=1}^{\infty} g_k(Tx) y_k = \sum_{k=1}^{\infty} f_k(x) y_k,$$

where $f_k \stackrel{\text{def}}{=} g_k \circ T \in X^*$. □

Def

A Banach space Y has the bounded approximation property (BAP) if there is $C > 0$ s.t. $\forall \varepsilon > 0 \forall y_1, \dots, y_n \in Y$ there exists $S \in F(Y)$ s.t. $\|S\| \leq C$ and $\|S y_k - y_k\| < \varepsilon$ for $k=1, \dots, n$. If we can take $C=1$, then Y is said to have the metric approximation property (MAP).

Proposition

If Y has (BAP), then $\overline{F(X, Y)} = K(X, Y)$ for any normed space X .

Proof

Take $T \in K(X, Y)$. Let $C > 0$ be as in the definition of (BAP) for Y . Take $\varepsilon > 0$. Then, for the unit ball $B \subset X$, we have $TB \subset \bigcup_{k=1}^n B_\varepsilon(y_k)$ for some $y_1, \dots, y_n \in Y$. By (BAP), there is $S \in F(Y)$ s.t. $\|S\| \leq C$ and $\|S y_k - y_k\| < \varepsilon$ for $k=1, \dots, n$. Then $ST \in F(X, Y)$. For any $x \in B$, there is k s.t. $\|Tx - y_k\| < \varepsilon$. Then $\|STx - S y_k\| \leq \|S\| \cdot \|Tx - y_k\| < C\varepsilon$.

Then we have

(9)

$$\|STx - Tx\| \leq \|STx - Sy_k\| + \|Sy_k - y_k\| + \|y_k - Tx\|$$

$$< C\varepsilon + \varepsilon + \varepsilon = (C+2)\varepsilon.$$

Since this is true for all $x \in B$, we conclude that $\|ST - T\| \leq (C+2)\varepsilon$.

As $\varepsilon > 0$ was arbitrary, it follows that $T \in \overline{F(X, Y)}$. □

