

Theorem

The following spaces have (MAP):

- (1) Hilbert spaces;
- (2) $L^p(\Omega, d\mu)$, $1 \leq p \leq \infty$, where $(\Omega, \mathcal{B}, \mu)$ is a σ -finite measure space;
- (3) $C(K)$, where K is a compact metric space.

Proof

- (1) If H is a Hilbert space and $y_1, \dots, y_n \in H$, then for the orthogonal projection $P: H \rightarrow \text{span}\{y_1, \dots, y_n\}$ we have $\|P\|=1$ (or $P=0$, if $y_k=0$ for all k) and $Py_k = y_k$.
- (2) Since μ is σ -finite, there is a probability measure ν on $(\Omega, \mathcal{B})$ equivalent to μ . Then $L^\infty(\Omega, \mu) = L^\infty(\Omega, \nu)$, while for $1 \leq p < \infty$ we have an isometric isomorphism $L^p(\Omega, d\mu) \cong L^p(\Omega, d\nu)$, $f \mapsto \left(\frac{d\mu}{d\nu}\right)^{1/p} f$ (exercise). Therefore without loss of generality we may assume that $\mu(\Omega)=1$.

Take $f_1, \dots, f_n \in L^p(\Omega, d\mu)$ and $\varepsilon > 0$. We can find simple functions $g_1, \dots, g_n \in L^p(\Omega, d\mu)$ s.t. $\|f_k - g_k\|_p < \frac{\varepsilon}{2}$ ($k=1, \dots, n$). The functions

g_n are measurable with respect to a finite subalgebra $\mathcal{A} \subset \mathcal{B}$. Consider the conditional expectation $S = \mathbb{E}(\cdot | \mathcal{A}) : L^p(\Omega, \mathcal{B}, \mu) \rightarrow L^p(\Omega, \mathcal{A}, \mu)$.

we know that $\|S\| = 1$. As $Sg_n = g_n$,

from $\|f_n - g_n\|_p < \frac{\varepsilon}{2}$ we then get

$$\|Sf_n - g_n\|_p = \|S(f_n - g_n)\|_p < \frac{\varepsilon}{2}.$$

$$\|Sf_n - f_n\|_p \leq \|Sf_n - g_n\|_p + \|g_n - f_n\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\text{Im } S = L^p(\Omega, \mathcal{A}, \mu)$ is f.d., we get the result.

(3) Assume K is a compact metric space.

Take $f_1, \dots, f_n \in C(K)$ and $\varepsilon > 0$.

Since K is compact, we can cover it by finitely many open balls $U_1, \dots, U_m$ s.t. if $x, y \in U_i$, then $|f_k(x) - f_k(y)| < \varepsilon$ for $k=1, \dots, n$.

Let $(p_k)_{k=1}^m$ be a partition of unity subordinate to the open covering $(U_k)_{k=1}^m$. For every $1 \leq k \leq m$, pick a point $x_k \in U_k$. Define $S : C(K) \rightarrow C(S)$ by

$$Sf = \sum_{k=1}^m f(x_k) p_k.$$

S is obviously of finite rank and

$$|(Sf)(x)| \leq \sum_{k=1}^m |f(x_k)| p_k(x)$$

$$\leq \|f\|_\infty \sum_{k=1}^m p_k(x) = \|f\|_\infty, \text{ so } \|Sf\|_\infty \leq \|f\|_\infty,$$

and therefore $\|S\| \leq 1$.

For $i \in \mathbb{N}$ and $x \in K$, we have

②

$$\begin{aligned} |(Sf_i)(x) - f_i(x)| &= \left| \sum_{k=1}^{\infty} f_i(x_k) p_k(x) - f_i(x) \sum_{k=1}^{\infty} p_k(x) \right| \\ &\leq \sum_{k=1}^{\infty} |f_i(x_k) - f_i(x)| p_k(x) \end{aligned}$$

Observe that if $x \notin U_k$, then $p_k(x) = 0$, while if $x \in U_k$, then $|f_i(x_k) - f_i(x)| < \varepsilon$.

Hence the above expression is smaller than

$$\sum_{k=1}^{\infty} \varepsilon p_k(x) = \varepsilon.$$

It follows that $\|Sf_i - f_i\|_{\infty} < \varepsilon$.

Corollary

If Y is one of the spaces in the theorem, then $\overline{F(X, Y)} = K(X, Y)$ for any normed space X .

Example

Consider the space $X = \ell^p$, $1 \leq p \leq \infty$.

Let $(a_n)_{n=1}^{\infty}$ be a sequence s.t. $a_n \xrightarrow{n \rightarrow \infty} 0$.

Define $T: X \rightarrow X$ by

$$T(x_1, x_2, \dots) = (a_1 x_1, a_2 x_2, \dots).$$

Then $\|T\| = \|(a_n)_n\|_{\infty} = \sup_n |a_n|$ (exercise)

Consider also $T_n: X \rightarrow X$ defined by

$$T_n(x_1, x_2, \dots) = (a_1 x_1, \dots, a_n x_n, 0, \dots).$$

Then $T_n \in F(X)$ and

$\|T - T_n\| = \sup_{k > n} |a_k| \xrightarrow{n \rightarrow \infty} 0$. Hence $T \in K(X)$.

Fact

(1) If H is an infinite-dimensional Hilbert space and $X = L(H)$, then $\overline{F(X)} \neq K(X)$ (Szankowski).

(2) For every $1 \leq p \leq \infty$, $p \neq 2$, there is a closed subspace $X \subset \ell^p$ s.t. $\overline{F(X)} \neq K(X)$.
(Enflo for $p > 2$, Szankowski for $1 \leq p < 2$).

Theorem

If $T \in K(X, Y)$ for some normed spaces X, Y , then $T^* \in K(Y^*, X^*)$.

If $K(X, Y) = \overline{F(X, Y)}$, then this is easy: if $T_n \in F(X, Y)$, $T_n \rightarrow T$, then $T_n^* \rightarrow T^*$ and $T_n^* \in F(Y^*, X^*)$ (exercise).

For Hilbert spaces, a similar proof shows that if $T \in K(H_1, H_2)$, then $T^* \in K(H_2, H_1)$.

In the general case we will use

Theorem (Arzelà - Ascoli)

For any compact metric space K , a subset $\Omega \subset C(K)$ is precompact if and only if it is

- pointwise bounded ($\sup_{f \in \Omega} |f(x)| < \infty \forall x \in K$);
- equicontinuous ($\forall \varepsilon > 0 \forall x \in K$ there is $\delta > 0$ s.t. if $d(x, y) < \delta$, then $|f(x) - f(y)| < \varepsilon \forall f \in \Omega$).

Proof of the theorem (T compact $\Rightarrow T^+$ compact) (3)

If $\bar{Y}$ is the completion of Y , then every $f \in Y^*$ extends by continuity to $\bar{Y}$, so that we can identify Y^* with $(\bar{Y})^*$. Therefore nothing changes if we replace Y by $\bar{Y}$. In other words, we may assume that Y is a Banach space. Then $K = \overline{TB_X} \subset Y$ is compact, where $B_X \subset X$ is the open unit ball.

Observe that we have a linear isometry $\text{Im } T^+ \rightarrow C(K)$, $T^+f = f \circ T \mapsto f|_K$,

$$\begin{aligned} \text{since } \|f|_K\|_\infty &= \sup_{y \in K} |f(y)| = \sup_{y \in TB_X} |f(y)| \\ &= \sup_{x \in B_X} |f(Tx)| = \sup_{x \in B_X} |(T^+f)(x)| = \|T^+f\|. \end{aligned}$$

We need to show that $T^+B_{Y^*}$ is precompact.

By the observation, this is equivalent to precompactness of $\mathcal{L} \stackrel{\text{def}}{=} \{f|_K : f \in B_{Y^*}\}$ in $C(K)$. By Arzelà-Ascoli, we need to check that $\mathcal{L}$ is pointwise bounded and equicontinuous.

For $y \in K$, we have $\sup_{f \in B_{Y^*}} |f(y)| = \|y\| < \infty$,

so $\mathcal{L}$ is pointwise bounded.

Given $y, y' \in K$ and $f \in B_{Y^*}$, we have

$$|f(y) - f(y')| \leq \|f\| \cdot \|y - y'\| \leq \|y - y'\|.$$

Thus, Ω is equicontinuous. $\square$