

Recall that the cokernel of a linear operator $T: X \rightarrow Y$ is defined by $\text{coker } T \stackrel{\text{def}}{=} Y / \text{Im } T$. ①

Def
A bounded linear operator $T: X \rightarrow Y$ between Banach spaces is called Fredholm if both $\ker T$ and $\text{coker } T$ are f.d. The (Fredholm) index of T is then defined by $\text{ind } T = \dim(\ker T) - \dim(\text{coker } T)$.

We denote by $\text{Fred}(X, Y)$ the set of Fredholm operators $X \rightarrow Y$. We write $\text{Fred}(X)$ for $\text{Fred}(X, X)$.

One's main motivation to study Fredholm operators is that, as we will see soon, if $A \in K(X)$, then $1 + A \in \text{Fred}(X)$ and $\text{ind}(1 + A) = 0$.

Def
Given a Banach space X , a closed subspace $Y \subset X$ is called complemented if there is a closed subspace $Z \subset X$ s.t. the map $Y \oplus Z \rightarrow X, (y, z) \mapsto y + z$, is an isomorphism. (Equivalently, $X = Y + Z \stackrel{\text{def}}{=} \{y + z: y \in Y, z \in Z\}$ and $Y \cap Z = \{0\}$.) We then write $X = Y \oplus Z$, and call Z a complement to Y .

If H is a Hilbert space, then every closed subspace $Y \subset H$ is complemented: we can take $Z = Y^\perp$. For general Banach spaces not all closed subspaces are complemented.

Lemma

Assume X is a Banach space and $Y \subset X$ is a closed subspace such that either Y or X/Y is f.d. Then Y is complemented. ($\dim X/Y$ is called the codimension of Y in X)

Proof

Assume Y is f.d. Choose a basis $y_1, \dots, y_n$ in Y and consider the dual basis $g_1, \dots, g_n$ in Y^* ($g_i(y_j) = \delta_{ij}$). Extend each g_i to a bounded linear functional f_i on X . Put $Z = \bigcap_{i=1}^n \ker f_i$. Then $Z \subset X$ is a closed subspace. If $y \in Y \cap Z$, then $y \in \bigcap_{i=1}^n \ker g_i = 0$, so $Y \cap Z = 0$. Given $x \in X$, let $y \stackrel{\text{def}}{=} \sum_{i=1}^n f_i(x) y_i$. Then $g_i(y) = f_i(x)$ for all i , so $x - y \in \bigcap_{i=1}^n \ker f_i = Z$. Thus, $Y + Z = X$.

Assume now that X/Y is f.d.

Choose a basis $e_1, \dots, e_n$ in X/Y , and then choose $x_i \in X$ s.t. $e_i = x_i + Y$, $i=1, \dots, n$.

Consider $Z = \text{span}\{x_1, \dots, x_n\}$. If $x \in Y \cap Z$, (2)
 then $x = a_1 x_1 + \dots + a_n x_n$ for some a_i and,
 on the one hand, $x + Y = 0$ in X/Y , while
 on the other hand, $x + Y = a_1 e_1 + \dots + a_n e_n$,
 so $a_1 = \dots = a_n = 0$. Thus, $x = 0$.

Finally, given $x \in X$, we have $x + Y = a_1 e_1 + \dots + a_n e_n$ for some $a_1, \dots, a_n$. Then
 $z \stackrel{\text{def}}{=} a_1 x_1 + \dots + a_n x_n \in Z$ and $x - z \in Y$,
 so $Y + Z = X$. □

Proposition

Assume X is a Banach space and $A \in K(X)$.
 Then $1 + A \in \text{Fred}(X)$.

Proof

Since $-A \in K(X)$, equally well we need
 to show that $1 - A \in \text{Fred}(X)$.

We have $\ker(1 - A) = \{x : Ax = x\}$.

Since A is compact and acts as the
 identity on $\{x : Ax = x\}$, the unit ball
 in this space is precompact. Hence this
 space is f.d., so $\ker(1 - A)$ is f.d.

Since $A^* \in K(X^*)$, for the same reason
 the space $\ker(1 - A^*)$ is f.d. We have

$$\left(\overline{\text{Im}(1 - A)}\right)^\perp = \{f \in X^* : f|_{\overline{\text{Im}(1 - A)}} = 0\}$$

$$\begin{aligned}
 &= \{ f \in X^*: f \circ (1-A) = 0 \} = \ker((1-A)^t) \\
 &= \ker(1-A^t). \text{ As } \left(\frac{X}{\operatorname{Im}(1-A)} \right)^* \simeq \left(\overline{\operatorname{Im}(1-A)} \right)^t, \\
 &\text{ we thus see that the space } \left(\frac{X}{\operatorname{Im}(1-A)} \right)^* \\
 &\text{ is f.d., hence } \frac{X}{\operatorname{Im}(1-A)} \text{ is f.d. as well.}
 \end{aligned}$$

To finish the proof it suffices to show that $\operatorname{Im}(1-A)$ is closed.

Choose a complement Y to $\ker(1-A)$. Then $T \stackrel{\text{def}}{=} (1-A)|_Y : Y \rightarrow X$ is injective and $\operatorname{Im} T = \operatorname{Im}(1-A)$. As a consequence of the open mapping theorem (recall our discussion of invertibility), $\operatorname{Im} T$ is closed if and only if there is $C > 0$ s.t. $\|Ty\| \geq C\|y\| \forall y \in Y$. Assume there is no such $C > 0$. Then there is a sequence of unit vectors $(y_n)_n$ in Y s.t. $Ty_n \rightarrow 0$. That is, $y_n - Ay_n \rightarrow 0$. As A is compact, by passing to a subsequence we may assume that $(Ay_n)_n$ is Cauchy, so $Ay_n \rightarrow x$ for some x . Since $\|y_n - Ay_n\| \rightarrow 0$, then we also have $y_n \rightarrow x$. It follows that $x \in Y$ and $Ax = x$, so that $Tx = 0$. As $\|x\| = \lim_{n \rightarrow \infty} \|y_n\| = 1$, this contradicts injectivity of T . $\square$

In the proof above we showed that $\text{Im}(I-A)$ is closed. This is true for all Fredholm operators:

Lemma

If X and Y are Banach spaces and $T \in \text{Fred}(X, Y)$, then $\text{Im} T \subset Y$ is closed.

Proof

We can find a f.d. subspace $Z \subset Y$ s.t. $\text{Im} T + Z = Y$ and $(\text{Im} T) \cap Z = \{0\}$ (recall the proof of complementability of subspaces of finite codimension). Consider the operator $S: X \oplus Z \rightarrow Y$, $S(x, z) \stackrel{\text{def}}{=} Tx + z$. Then S is bounded and surjective, so by the open mapping theorem it is open. Since $\{(x, z): z \neq 0\} \subset X \oplus Z$ is open, its image $\{Tx + z: z \neq 0\} \subset Y$ is open as well, hence its complement $\text{Im} T$ is closed. $\square$

It is often convenient to think of a Fredholm operator in terms of matrices of operators. For this, let us first consider a more general situation.

Assume X and Y are Banach spaces and we have decompositions $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$ for some closed subspaces $X_i \subset X$

and $Y_i \subset Y$. Denote by $p_i: X \rightarrow X_i$
 and $q_j: Y \rightarrow Y_j$ the maps $(x_1, x_2) \mapsto x_i$,
 $(y_1, y_2) \mapsto y_j$, resp. For every $T \in L(X, Y)$,
 consider the operators

$$T_{ij} \stackrel{\text{def}}{=} q_i T|_{X_j}: X_j \rightarrow Y_i$$

Then $T_{ij} \in L(X_j, Y_i)$ and

$$T(x_1, x_2) = (T_{11}x_1 + T_{12}x_2, T_{21}x_1 + T_{22}x_2),$$

so T can be represented by the matrix $\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$.

Lemma

An operator $T \in L(X, Y)$ is Fredholm if
 and only if there are decompositions
 $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ s.t. the spaces
 X_2 and Y_2 are f.d. and T has the form
 $\begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix}$, with $T_{11}: X_1 \rightarrow Y_1$ invertible.
 Moreover, then $\text{ind } T = \dim X_2 - \dim Y_2$.

Proof

If $T \in \text{Fred}(X, Y)$, then we take
 $X_2 = \ker T$, $Y_1 = \text{Im } T$ and arbitrary
 complements X_1 to X_2 and Y_2 to Y_1 .

Conversely, if $T = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix}$ for

some decompositions $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$
 with f.d. X_2 and Y_2 and invertible T_{11} ,
 then $\ker T = X_2$ and $\text{Im } T = Y_1$, so $Y / \text{Im } T \cong Y_2$. ◻