

If $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ with respect to ①
 decompositions $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$,
 then $X^* = X_1^* \oplus X_2^*$, $Y^* = Y_1^* \oplus Y_2^*$, where we
 view $f \in X_i^*$ as a linear functional
 on X by $(x_1, x_2) \mapsto f(x_i)$ and similarly
 for $g \in Y_j^*$, and $T^t = \begin{pmatrix} T_{11}^t & T_{21}^t \\ T_{12}^t & T_{22}^t \end{pmatrix}$
 (exercise). As an immediate application

we get
corollary

If $T \in \text{Fred}(X, Y)$, then $T^t \in \text{Fred}(Y^*, X^*)$
 and $\text{ind } T^t = -\text{ind } T$.

Proof

If $T = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix}$, with invertible T_{11} ,
 then $T^t = \begin{pmatrix} T_{11}^t & 0 \\ 0 & 0 \end{pmatrix}$ and T_{11}^t is invertible.
 (exercise) □

The following result gives a very useful
 characterization of Fredholm operators.

Theorem (Atkinson)

Assume X and Y are Banach spaces
 and $T \in L(X, Y)$. Then the following
 conditions are equivalent:

(i) $T \in \text{Fred}(X, Y)$;

(2) there is $S \in \mathcal{K}(Y, X)$ s.t.
 $1 - ST \in \mathcal{F}(X)$ and $1 - TS \in \mathcal{F}(Y)$;

(3) there is $S \in \mathcal{K}(Y, X)$ s.t.
 $1 - ST \in \mathcal{K}(X)$, $1 - TS \in \mathcal{K}(Y)$.

Proof

(1) $\Rightarrow$ (2)

Consider decompositions $X = X_1 \oplus X_2$ and
 $Y = Y_1 \oplus Y_2$ s.t. X_2 and Y_2 are f.d.,

$T = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix}$ and T_{11} is invertible.

Define $S \stackrel{\text{def}}{=} \begin{pmatrix} T_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}$. Then

$1 - ST = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $1 - TS = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

(2) $\Rightarrow$ (3)

Obvious, as finite rank operators are compact.

(3) $\Rightarrow$ (1)

If $1 - ST$ and $1 - TS$ are compact, then
the operators $ST = 1 - (1 - ST)$ and $TS = 1 - (1 - TS)$
are Fredholm. Hence $\ker ST$ is f.d. and the space $\text{Im } T \supset \text{Im } TS$
is of finite codimension in Y . Thus,
 T is Fredholm. $\square$

Recall that the index of $T \in \text{Fred}(X, Y)$ is defined by

$$\text{ind } T = \dim(\ker T) - \dim(\text{coker } T).$$

Theorem

Assume X, Y, Z are Banach spaces and $T \in \text{Fred}(X, Y)$, $S \in \text{Fred}(Y, Z)$. Then

$ST \in \text{Fred}(X, Z)$ and

$$\text{ind}(ST) = \text{ind } S + \text{ind } T.$$

For the proof it is convenient to develop a general framework.

By an exact sequence of vector spaces one means a sequence $(X_n)_n$ of vector spaces, where n runs through an interval of $\mathbb{Z}$, together with linear operators $d_n: X_n \rightarrow X_{n+1}$ s.t. $\text{Im } d_n = \ker d_{n+1}$ for all n .

Proposition

Assume $0 \rightarrow X_0 \xrightarrow{d_0} X_1 \rightarrow \dots \xrightarrow{d_{n-1}} X_n \rightarrow 0$ is an exact sequence of f.d. vector spaces.

Then $\sum_{k=0}^n (-1)^k \dim X_k = 0$.

Proof

we will prove this by induction on n .

The cases $n=0$ and $n=1$ are immediate:

in first case $X_0=0$, in the second $\dim X_0 = \dim X_1$.

The first interesting case is $n=2$.

$$0 \rightarrow X_0 \xrightarrow{d_0} X_1 \xrightarrow{d_1} X_2 \rightarrow 0$$

Then $d_0: X_0 \rightarrow X_1$ is injective and d_1 defines an isomorphism $X_1/d_0(X_0) \cong X_2$. Thus,

$$\begin{aligned} \dim X_1 &= \dim d_0(X_0) + \dim X_1/d_0(X_0) \\ &= \dim X_0 + \dim X_2. \end{aligned}$$

For the induction step consider

$$0 \rightarrow X_0 \xrightarrow{d_0} X_1 \xrightarrow{d_1} \dots \rightarrow X_{n+1} \rightarrow 0 \quad (n \geq 2)$$

Then we have two exact sequences

$$0 \rightarrow X_0 \xrightarrow{d_0} X_1 \xrightarrow{d_1} d_1(X_1) \rightarrow 0,$$

$$0 \rightarrow d_1(X_1) \rightarrow X_2 \rightarrow \dots \rightarrow X_{n+1} \rightarrow 0.$$

$$\text{Hence } \dim X_0 - \dim X_1 + \dim d_1(X_1) = 0$$

and, by the inductive assumption,

$$\dim d_1(X_1) - \dim X_2 + \dots + (-1)^n \dim X_{n+1} = 0.$$

Subtracting these two equalities from each other we get $\sum_{n=0}^{n+1} (-1)^k \dim X_n = 0$. □

Proof of the theorem

Given Fredholm operators $X \in \mathcal{F}$ and $Y \in \mathcal{F}$, we claim that we have the following exact sequence of vector spaces:

$$0 \rightarrow \ker T \xrightarrow{d_0} \ker ST \xrightarrow{d_1} \ker S \quad (3)$$

$$x \mapsto Tx$$

$$\xrightarrow{d_2} \operatorname{coker} T \xrightarrow{d_3} \operatorname{coker}(ST) \xrightarrow{d_4} \operatorname{coker} S \rightarrow 0$$

$$y + \operatorname{Im} T \mapsto Sy + \operatorname{Im} ST$$

where all unspecified maps are either obvious embedding or quotient maps.

Let us check, for example, exactness at $\operatorname{coker} T$ (which is the least obvious), that is, $\operatorname{Im} d_2 = \ker d_3$.

Given $y \in \ker S$, we have $d_3 d_2 y = d_3(y + \operatorname{Im} S) = Sy + \operatorname{Im} ST = 0$. Thus, $\operatorname{Im} d_2 \subset \ker d_3$. To prove the equality, take $y \in Y$ s.t. $d_3(y + \operatorname{Im} T) = 0$. This means that $Sy \in \operatorname{Im} ST$, so $Sy = STx$ for some $x \in X$. Then $S(y - Tx) = 0$, so $y - Tx \in \ker S$ and hence $y + \operatorname{Im} T = d_2(y - Tx)$.

Exactness implies that the spaces $\ker ST$ and $\operatorname{coker} ST$ are f.d., since

$$\begin{aligned} \dim(\ker ST) &= \dim d_0(\ker T) + \dim(\ker ST / \operatorname{Im} d_0) \\ &= \dim d_0(\ker T) + \dim \ker d_1(\ker ST) \\ &\leq \dim(\ker T) + \dim(\ker S), \end{aligned}$$

and similarly for $\dim(\operatorname{coker} ST)$.

By the proposition we then get

$$\dim(\ker T) - \dim(\ker ST) + \dim(\ker S)$$

$$- \dim(\operatorname{coker} T) + \dim(\operatorname{coker} ST) - \dim(\operatorname{coker} S) = 0,$$

$$\text{that is, } \operatorname{ind} T + \operatorname{ind} S - \operatorname{ind}(ST) = 0.$$

□