

Theorem

Assume X, Y are Banach spaces. Then $\text{Fred}(X, Y)$ is an open subset of $L(X, Y)$ and the map $\text{ind}: \text{Fred}(X, Y) \rightarrow \mathbb{Z}$ is continuous (thus, locally constant).

Proof

Take $T \in \text{Fred}(X, Y)$. Consider decompositions

$X = X_1 \oplus X_2, Y = Y_1 \oplus Y_2$ s.t. X_2, Y_2 are f.d.

and $T = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix}$, with $T_{11} \in L(X_1, Y_1)$ invertible.

Take an arbitrary $S \in L(X, Y)$ and consider its matrix $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$. Then, recall,

$S_{11} = q_1 S|_{X_1}$, where $q_1: Y \rightarrow Y_1$ is the projection map $(y_1, y_2) \mapsto y_1$. Hence

$$\|S_{11} - T_{11}\| = \|q_1(S - T)|_{X_1}\| \leq \|q_1\| \|S - T\|.$$

Therefore if $\|S - T\| < \frac{1}{\|q_1\| \cdot \|T_{11}^{-1}\|}$,

then $\|S_{11} - T_{11}\| < \frac{1}{\|T_{11}^{-1}\|}$ and therefore S_{11} is invertible. Consider the operator

$$R \stackrel{\text{def}}{=} \begin{pmatrix} S_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}; Y \rightarrow X. \text{ Then}$$

$$I - RS = \begin{pmatrix} 0 & S_{11}^{-1} S_{12} \\ 0 & I \end{pmatrix} \in \mathcal{F}(X). \text{ Similarly,}$$

$1 - SR \in F(Y)$. Hence S is Fredholm.

This proves the first part of the theorem.

We claim that we also have $\text{ind } S = \text{ind } T$, which will prove the second part as well.

we have

$$\text{ind } T = \dim X_2 - \dim Y_2 = -\text{ind } R.$$

hence it suffices to show that

$\text{ind } S = -\text{ind } R$. Since $\text{ind}(RS) = \text{ind } R + \text{ind } S$, we just need to show that $\text{ind}(RS) = 0$.

Since $1 - RS \in F(X)$, for this, in turn, it suffices to show that if $A \in F(X)$, then

$$\text{ind}(1 + A) = 0.$$

Write $A = \sum_{i=1}^n f_i(\cdot) x_i$.

Consider the space $\bigcap_{i=1}^n \ker f_i$ of finite codimension in X , and choose a subspace $\tilde{X}_1 \subset \bigcap_{i=1}^n \ker f_i$ of finite codimension complementary to $(\bigcap_{i=1}^n \ker f_i) \cap \text{span}\{x_i\}$.

Then choose a f.d. complementary subspace $\tilde{X}_2 \subset X$ containing $x_1, \dots, x_n$. Then

$(1+A)|_{\tilde{X}_1} = 1$ and $(1+A)\tilde{X}_2 \subset \tilde{X}_2$. Hence

$$\text{ind}(1+A) = \text{ind}\left((1+A)|_{\tilde{X}_2}\right) = \dim \tilde{X}_2 - \dim \tilde{X}_2 = 0.$$

exercise

②

Corollary

If $T \in \text{Fred}(X, Y)$ and $A \in K(X, Y)$, then $T+A \in \text{Fred}(X, Y)$ and $\text{ind}(T+A) = \text{ind} T$.

Proof

Choose $S \in L(Y, X)$ s.t. $I - TS \in K(Y)$ and $I - ST \in K(X)$. Then

$$I - (T+A)S = (I - TS) + AS \in K(Y)$$

and similarly $I - S(T+A) \in K(X)$. Hence $T+A \in \text{Fred}(X, Y)$ by Atkinson's theorem.

Consider the operators $T_t = T + tA$, $0 \leq t \leq 1$.

Then $T_t \in \text{Fred}(X, Y)$ and the map $t \mapsto \text{ind}(T_t)$ is continuous, hence constant, so $\text{ind} T = \text{ind}(T+A)$. $\square$

Corollary

For any $T \in L(X, Y)$, the following conditions are equivalent:

(1) $T \in \text{Fred}(X, Y)$ and $\text{ind} T = 0$;

(2) there is $A \in K(X, Y)$ s.t. $T+A$ is invertible;

(3) there is $A \in F(X, Y)$ s.t. $T+A$ is invertible.

Proof

(1) $\Rightarrow$ (3)

Write again $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$ s.t.
 X_2, Y_2 are f.d., $T = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix}$ and T_{11} is
invertible. As $\text{ind } T = 0$ by assumption,
we have $\dim X_2 = \dim Y_2$. Choose and
linear isomorphism $S: X_2 \rightarrow Y_2$ and
consider $A \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}$. Then $A \in F(X, Y)$
and $T + A$ is invertible.

(3) $\Rightarrow$ (2)

Clear, as $F(X, Y) \subset K(X, Y)$.

(2) $\Rightarrow$ (1)

If $T + A$ is invertible, then obviously
 $T + A \in \text{Fred}(X, Y)$ and $\text{ind}(T + A) = 0$.

Then, by the previous corollary,

$T = (T + A) - A \in \text{Fred}(X, Y)$ and $\text{ind } T = \text{ind}(T + A) = 0$. □

Therefore the index measures in
some sense how far we are from
perturbing a Fredholm operator by
a compact operator to get an invertible one.

Example (Toeplitz operators)

Consider ℓ^2 and the shift

$$u: \ell^2 \rightarrow \ell^2, \quad u(a_0, a_1, \dots) = (0, a_0, a_1, \dots).$$

(3)

Then $u \in \text{Fred}(\ell^2)$ and
 $\text{ind } u = -1$.

The adjoint of u is

$$u^*: \ell^2 \rightarrow \ell^2, \quad u^*(a_0, a_1, \dots) = (a_1, a_2, \dots).$$

Then $\text{ind}(u^*) = 1$.

More generally, for any $u \geq 0$,

$$\text{ind}(u^n) = -n, \quad \text{ind}((u^*)^n) = n.$$

The operators $u^n, (u^*)^n$ are examples of Toeplitz operators.

Consider the space $L^2(\mathbb{T})$ with respect to the normalized Lebesgue measure $\lambda_{\mathbb{T}}$ on $\mathbb{T}$, so that $\int_{\mathbb{T}} f d\lambda_{\mathbb{T}} = \int_0^{2\pi} f(e^{it}) dt$.

It has an orthonormal basis $(e_n)_{n \in \mathbb{Z}}$,

where $e_n(z) = z^n$. Consider the Hardy

Space $H^2 \stackrel{\text{def}}{=} \text{span}\{e_n : n \geq 0\} \subset L^2(\mathbb{T})$.

Let $P: L^2(\mathbb{T}) \rightarrow H^2$ be the orthogonal projection.

Every function $f \in C(\mathbb{T})$ defines an operator A_f of multiplication by f on $L^2(\mathbb{T})$: $A_f g = fg$. We then define

$T_f: H^2 \rightarrow H^2$ by $T_f \stackrel{\text{def}}{=} PA_f|_{H^2}$, so

$T_f g = P(fg)$. T_f is called the Toeplitz operator with symbol f .

If we identify H^2 with ℓ^2 , then

$$u = T e_1, \quad u^* = T e_{-1}.$$

We have $A_f A_g = A_{fg}$. A similar equality is no longer true for Toeplitz operators, but we have

Lemma

For any $f, g \in C(\mathbb{T})$, we have

$$T_f T_g - T_{fg} \in K(H^2).$$