

Proof

①

Since the map $f \mapsto T_f$ is linear and $\|T_f\| \leq \|A_f\| = \|f\|$, it suffices to prove the lemma for $f = e_n, g = e_m$, as the functions e_k span a dense subspace of $C(\mathbb{T})$ by Weierstrass' theorem.

If n, m are of the same sign, then $T_e T_{e_m} = T_{e_{n+m}}$. It remains to consider four cases, namely, in terms of the shift $u = T_e$, on ℓ^2 we need to show that

the following operators are compact:

$$(1) \quad u^n (u^*)^m - u^{n-m} \quad (n \geq m \geq 0);$$

$$(2) \quad u^n (u^*)^m - (u^*)^{m-n} \quad (m \geq n \geq 0);$$

$$(3) \quad (u^*)^m u^n - u^{n-m} \quad (n \geq m \geq 0);$$

$$(4) \quad (u^*)^m u^n - (u^*)^{m-n} \quad (m \geq n \geq 0).$$

In fact, we will show that these operators have finite rank.

We will only consider case (1). Case (3) is similar, and the remaining two reduce to these by taking adjoints.

Thus, assume $n \geq m \geq 0$.

$$\begin{aligned} u^n (u^*)^m (a_0, a_1, \dots) &= u^n (a_m, a_{m+1}, \dots) \\ &= (\underbrace{0, \dots, 0}_n, a_m, a_{m+1}, \dots), \end{aligned}$$

$$u^{n-m}(a_0, a_1, \dots) = (0, \dots, 0, a_0, a_1, \dots).$$

$\underbrace{\hspace{2cm}}_{n-m}$

Therefore

$$\begin{aligned} (u^n(u^*)^m - u^{n-m})(a_0, a_1, \dots) \\ = (0, \dots, 0, -a_0, -a_1, \dots, -a_{m-1}, 0, \dots) \end{aligned}$$

$\underbrace{\hspace{2cm}}_{n-m}$

so $u^n(u^*)^m - u^{n-m} \in F(\ell^2)$. □

Corollary

Assume $f \in C(\Pi)$. If $0 \notin f(\Pi)$, then

T_f is Fredholm.

(In fact, it is "if and only if".)

Proof

As $0 \notin f(\Pi)$, we have $\frac{1}{f} \in C(\Pi)$.

Then $T_f T_{\frac{1}{f}} - I = T_f T_{\frac{1}{f}} - T_1 \in K(\ell^2)$,

$T_{\frac{1}{f}} T_f - I = T_{\frac{1}{f}} T_f - T_1 \in K(\ell^2)$, hence

T_f is Fredholm by Atkinson's theorem. □

Thus, if $0 \notin f(\Pi)$, then we have a well-defined index $\text{ind } T_f$. What is it?

Def

Given metric spaces X and Y , two continuous maps $f, g: X \rightarrow Y$ are called homotopic if there exist

continuous maps $f_t: X \rightarrow Y$ ($0 \leq t \leq 1$) ②
 s.t. $f_0 = f$, $f_1 = g$ and the map
 $[0, 1] \times X \rightarrow Y$, $(t, x) \mapsto f_t(x)$, is
 continuous. Then $(f_t)_t$ is called
 a homotopy between f and g .

Fact

Any continuous map $f: \mathbb{T} \rightarrow \mathbb{C}^* = \{z \in \mathbb{C} : z \neq 0\}$
 is homotopic to $e_n: \mathbb{T} \rightarrow \mathbb{C}^*$ ($e_n(z) = z^n$)
 for a uniquely defined $n \in \mathbb{Z}$. The number
 n is called the winding number of f
 and is denoted by $\text{wind } f$.

(Sketch: one can find a continuous
 function $F: [0, 1] \rightarrow \mathbb{R}$ s.t.

$f(e^{2\pi i t}) = |f(e^{2\pi i t})| e^{2\pi i F(t)}$ ($0 \leq t \leq 1$),
 and F is uniquely determined up to
 adding an integer. As $e^{2\pi i F(1)} = e^{2\pi i F(0)}$,
 we get a number $n \stackrel{\text{def}}{=} F(1) - F(0) \in \mathbb{Z}$.

Define $f_s(e^{2\pi i t}) \stackrel{\text{def}}{=} |f(e^{2\pi i t})|^s e^{2\pi i ((1-s)t + sF(t))}$.

Then $(f_s)_{0 \leq s \leq 1}$ is a homotopy between
 e_n and f .)

Index theorem for Toeplitz operators

If $f \in C(\Pi)$, $0 \notin f(\Pi)$, then
 $\text{ind } T_f = -\text{wind } f$.

Proof

Let $n = \text{wind } f$. Choose a homotopy
 $(f_t)_{0 \leq t \leq 1}$ between f and e_n .
Then $t \mapsto \text{ind } T_{f_t}$ is continuous,
hence constant. Therefore
 $\text{ind } T_f = \text{ind } T_{e_n} = -n = -\text{wind } f. \quad \square$

③

We now apply what we have learnt to analyze spectra of compact operators.

Theorem

Assume X is an infinite-dimensional complex Banach space, $T \in K(X)$. Then

(1) $\sigma(T) = \{0\} \cup \sigma_p(T)$;

(2) the set $\sigma_p(T) \setminus \{0\}$ is either finite or a sequence converging to 0;

(3) the multiplicity of every eigenvalue $\lambda \in \sigma_p(T) \setminus \{0\}$ is finite and equal to $\dim \ker(T - \lambda I) = \dim \operatorname{coker}(T - \lambda I) = \dim \ker(T^t - \lambda I) = \dim \operatorname{coker}(T^t - \lambda I)$.

Proof

(1) Since X is infinite-dimensional, any open ball in X is not pre-compact, hence any invertible operator $X \rightarrow X$ is not compact. Thus, $0 \in \sigma(T)$.

Take $\lambda \notin \sigma_p(T)$, $\lambda \neq 0$. Then $T - \lambda I$ is Fredholm and $\operatorname{ind}(T - \lambda I) = 0$. As $\ker(T - \lambda I) = 0$ by assumption, we conclude that $\operatorname{coker}(T - \lambda I) = 0$, that is, $\operatorname{Im}(T - \lambda I) = X$. Thus, $T - \lambda I$ is invertible, so $\lambda \notin \sigma(T)$. Therefore $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$.

2) Equivalently, we need to show that for every $\varepsilon > 0$ the set $\mathcal{B}_p(T) \cap \{\lambda: |\lambda| \geq \varepsilon\}$ is finite. Assume this is not the case, so there are $\varepsilon > 0$ and different $\lambda_n \in \mathcal{B}_p(T)$ ($n \geq 1$) s.t. $|\lambda_n| \geq \varepsilon$. Choose unit vectors x_n s.t. $Tx_n = \lambda_n x_n$.

Consider $X_n = \text{span}\{x_1, \dots, x_n\}$. By Riesz' lemma we can find unit vectors $y_n \in X_n$ s.t. $d(y_n, X_{n-1}) > \frac{1}{2}$. We claim that, although the sequence $\left(\frac{y_n}{\lambda_n}\right)_{n=1}^\infty$ is bounded, the sequence $\left(T\left(\frac{y_n}{\lambda_n}\right)\right)_n$ does not have a Cauchy subsequence, contradicting compactness of T .

Indeed, write

$$y_n = a_n x_n + z_n \quad (a_n \in \mathbb{C}^x, z_n \in X_{n-1})$$

$$\begin{aligned} \text{Then } T\left(\frac{y_n}{\lambda_n}\right) &= a_n x_n + T\left(\frac{z_n}{\lambda_n}\right) \\ &= y_n + \underbrace{\left(T\left(\frac{z_n}{\lambda_n}\right) - z_n\right)}_{\substack{\uparrow \\ X_{n-1}}} \end{aligned}$$

Hence, for any $k < n$,

$$\begin{aligned} \left\| T\left(\frac{y_n}{\lambda_n}\right) - T\left(\frac{y_k}{\lambda_k}\right) \right\| &\geq d\left(T\left(\frac{y_n}{\lambda_n}\right), X_{n-1}\right) \\ &= d(y_n, X_{n-1}) > \frac{1}{2}, \text{ proving} \\ &\text{our claim.} \end{aligned}$$

(3) For every $\lambda \in \sigma_p(T) \setminus \{0\}$, the operators $T - \lambda I$ and $T^* - \lambda I$ are Fredholm, of index 0, so

$$\dim \ker(T - \lambda I) = \dim \operatorname{coker}(T - \lambda I) < \infty,$$

$$\dim \ker(T^* - \lambda I) = \dim \operatorname{coker}(T^* - \lambda I) < \infty.$$

We also have

$$\dim \operatorname{coker}(T - \lambda I) = \dim \ker(T^* - \lambda I)$$

$$\left(\text{as } \left(\frac{X}{\operatorname{Im}(T - \lambda I)} \right)^* \simeq \left(\operatorname{Im}(T - \lambda I) \right)^\perp \subseteq \ker(T^* - \lambda I) \right).$$

Historically, the first major results of this type (for particular spaces and operators) were the following:

Fredholm alternatives

Assume X is a Banach space (complex or real), $T \in K(X)$ and $\lambda \neq 0$ is a scalar.

Then

(1) the equation $Tx - \lambda x = 0$ in $x \in X$ has a nonzero solution if and only if the equation $T^*f - \lambda f = 0$ has a nonzero solution in X^* ;

(2) for $y \in X$, the equation $Tx - \lambda x = y$ has a ~~unique~~ solution in X if and only if $f(y) = 0$ for all $f \in X^*$ s.t.

$$T^*f = \lambda f.$$

Proof

(1) We know that $\dim \ker(T - \lambda I) = \dim \ker(T^* - \lambda I)$
by part (3) of the previous theorem.

(2) We know that $\text{Im}(T - \lambda I)$ is closed
and $\left(\overline{\text{Im}(T - \lambda I)} \right)^\perp \subseteq \ker(T^* - \lambda I)$.

hence

$$y \in \text{Im}(T - \lambda I) \Leftrightarrow y + \text{Im}(T - \lambda I) = 0 \text{ in } \overline{\text{Im}(T - \lambda I)}$$

$$\Leftrightarrow F(y + \text{Im}(T - \lambda I)) = 0 \text{ for all } F \in \left(\overline{\text{Im}(T - \lambda I)} \right)^\perp$$

$$\Leftrightarrow F(y) = 0 \text{ for all } F \in \ker(T^* - \lambda I).$$

□