

Let us also note that

$$b(u) = b(u^*) = \{z \in \mathbb{C} : |z| \leq 1\} = \overline{\mathbb{D}}.$$

For $|\lambda| \leq 1$ let us try to solve

$$u^*(a_0, \dots) = \lambda(a_0, \dots).$$

$$u^*(a_1, a_2, \dots)$$

This means that $a_n = \lambda a_{n-1}$ for $n \geq 1$.

Therefore $(1, \lambda, \lambda^2, \dots)$ is a solution if $|\lambda| < 1$.

Hence $b_p(u^*) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, and

$$\text{then } b(u^*) \supset \overline{b_p(u^*)} = \overline{\mathbb{D}}.$$

On the other hand, $\|u^n\| = 1 \quad \forall n \geq 0$,

so $r(u^*) = r(u) = 1$. Hence $b(u^*) = \overline{\mathbb{D}}$,

and then $b(u) = \{\bar{\lambda} : \lambda \in b(u^*)\} = \overline{\mathbb{D}}.$

Spectral theorem for compact self adjoint operators ②

Let H be a Hilbert space, $A \in K(H)$, $A = A^*$. We already know:

- $\sigma(A) \subset [-\|A\|, \|A\|]$, with one of the end points included in the spectrum (since $\sigma(A) \neq \emptyset$);
- $\sigma(A) \setminus \{0\} = \sigma_p(A) \setminus \{0\}$, and $0 \in \sigma(A)$ if $\dim H = \infty$;
- $\sigma_p(A) \setminus \{0\}$ is either finite or a sequence converging to 0.

Spectral theorem (Hilbert-Schmidt)
With A as above, there is an orthonormal basis consisting of eigenvectors of A .

For the proof recall two properties familiar from linear algebra.

The first is that if $A = A^*$, $Ax = \lambda x$, $Ay = \mu y$ for $\lambda \neq \mu$, then $x \perp y$. Indeed,

$$\lambda(x, y) = (Ax, y) = (x, Ay) = \mu(x, y),$$

hence $(x, y) = 0$.

The second is

Lemma

If H is a Hilbert space, $A \in \mathcal{K}(H)$ and $K \subset H$ is closed A -invariant subspace (that is, $AK \subset K$), then $K^\perp$ is A^* -invariant.

Proof

Take $x \in K^\perp$. We need to show that $A^*x \in K^\perp$, that is, $(A^*x, y) = 0 \quad \forall y \in K$.

We have:

$$(A^*x, y) = \underbrace{(x, A)}_{\substack{\uparrow \\ K^\perp}} \underbrace{y}_{\substack{\uparrow \\ K}} = 0.$$

□

Proof of the theorem

We will give two proofs.

1st proof

For every $\lambda \in \sigma_p(A) \setminus \{0\}$ choose an orthonormal basis in $\ker(A - \lambda I)$.

Taking the union of all these bases we get an orthonormal system in H consisting of eigenvectors of A . Let K be its closed span. Then $AK \subset K$.

Hence $AK^\perp \subset K^\perp$. The operator $A|_{K^\perp}$ is self-adjoint, compact, with

no nonzero eigenvalues. Hence

$$\|A|_{K^\perp}\| = s(A|_{K^\perp}) = 0, \text{ so } A|_{K^\perp} = 0.$$

Choose an orthonormal basis in $K^\perp$. ③

Together with our chosen orthonormal basis in K this gives an orthonormal basis in H consisting of eigenvectors of A .

2nd proof (not relying on earlier results)

The crucial part in the previous proof is the claim that if A is compact, $A = A^*$ and A has no nonzero eigenvalues, then $A = 0$. We will prove this claim without using earlier results.

Equivalently, we need to show that if $A = A^*$, A is compact and $A \neq 0$, then there is a nonzero eigenvalue.

First of all let us show that

$$\|A\| = \sup_{\|x\|=1} |(Ax, x)|$$

(This does not need compactness, only $A = A^*$.)

Indeed, the inequality $\geq$ holds by the

Cauchy-Schwarz inequality:

$$|(Ax, x)| \leq \|Ax\| \cdot \|x\| \leq \|A\| \cdot \|x\|^2 = \|A\|.$$

For the opposite inequality, denote by M the right-hand side. Then $|(Ax, x)| \leq M \|x\|^2$ for all x . Take x s.t. $\|x\|=1$, and

assume $Ax \neq 0$. Then, for $y = \frac{Ax}{\|Ax\|}$ we have

$$4\|Ax\|^2 = (A(x+y), x+y) - (A(x-y), x-y) \\ \text{(since } (Ax, y) = \|Ax\| \text{)}$$

$$\leq M\|x+y\|^2 + M\|x-y\|^2 \\ = 2M(\|x\|^2 + \|y\|^2) \\ = 4M,$$

so $\|Ax\| \leq M$, and hence $\|A\| \leq M$.

It follows that there exists a sequence of unit vectors x_n s.t. $|(Ax_n, x_n)| \rightarrow \|A\|$. By passing to a subsequence we may assume that $(Ax_n, x_n) \rightarrow \alpha$, where $\alpha = \|A\|$ or $\alpha = -\|A\|$. Since A is compact, by passing to a subsequence we may assume that $Ax_n \rightarrow y$ for some y . We have

$$\|Ax_n - \alpha x_n\|^2 = (Ax_n - \alpha x_n, Ax_n - \alpha x_n) \\ = \|Ax_n\|^2 + \alpha^2 - 2\alpha(Ax_n, x_n) \\ \leq 2\alpha^2 - 2\alpha(Ax_n, x_n)$$

$$\xrightarrow{n \rightarrow \infty} 2\alpha^2 - 2\alpha^2 = 0.$$

As $Ax_n \rightarrow y$, we conclude that $\alpha x_n \rightarrow y$.

Then $\|y\| = |\alpha| = \|A\|$ and

$$Ay = \lim_{n \rightarrow \infty} A(\alpha x_n) = \alpha \lim_{n \rightarrow \infty} Ax_n = \alpha y,$$

so α is a nonzero eigenvalue of A . $\square$

Remark

(4)

Using the 2nd proof we can reproove some of the thing we already know:

- 1) The proof shows that if $A = A^*$, then there is a sequence of unit vectors x_n s.t.
 $\|Ax_n - \alpha x_n\| \rightarrow 0$, where $\alpha = \|A\|$ or $\alpha = -\|A\|$.

It follows that $\alpha \in \sigma(A)$, so $r(A) \geq \|A\|$.

- 2) If $A = A^* \in K(H)$, then $\sigma_p(A) \setminus \{0\}$ is either finite or a sequence converging to 0, and the multiplicity of every $\lambda \in \sigma_p(A) \setminus \{0\}$ is finite. Indeed, let $(e_i)_i$ be an orthonormal basis consisting of eigenvectors.

If the claim is not true, we can find choose a sequence of different vectors x_n out of $(e_i)_i$ s.t. $Ax_n = \lambda_n x_n$, $|\lambda_n| \geq \varepsilon$ for some $\varepsilon > 0$. Since $x_n \perp x_m$ for $n \neq m$, we then have

$$\begin{aligned}\|Ax_n - Ax_m\|^2 &= \|\lambda_n x_n - \lambda_m x_m\|^2 \\ &= |\lambda_n|^2 \|x_n\|^2 + |\lambda_m|^2 \|x_m\|^2 \\ &= |\lambda_n|^2 + |\lambda_m|^2 \geq 2\varepsilon^2.\end{aligned}$$

Hence $(Ax_n)_n$ does not have a Cauchy subsequence, contradicting compactness of A .

