

The spectral theorem for compact self-adjoint operators gives also information about non-self adjoint ones. ①

Assume H is a Hilbert space, $T \in K(H)$. Then $T^*T \in K(H)$ and T^*T is self-adjoint.

Moreover, T^*T is positive: a self-adjoint operator A is called positive if

$$(Ax, x) \geq 0 \quad \forall x \in H,$$

in which case we write $A \geq 0$.

Let us list the square roots of the nonzero eigenvalues of T^*T as

$$s_1(T) \geq s_2(T) \geq \dots,$$

counting with multiplicities. The sequence $(s_n(T))_n$ either stops after a certain n or converges to 0. The numbers $s_n(T)$ are called the (nonzero) singular values of T .

We want to make sense of $(T^*T)^{1/2}$. For this we can use the spectral theorem: find an orthonormal basis $(e_i)_i$ s.t. $T^*T e_i = \lambda_i e_i$. Then define $(T^*T)^{1/2} e_i = \lambda_i^{1/2} e_i$.

In fact, $(T^*T)^{1/2}$ can be defined without compactness and the spectral theorem:

Theorem (Functional calculus for self-adjoint operators)

Assume $A = A^* \in L(H)$ for a Hilbert space H .

Then there is a unique bounded linear

map $C(\sigma(A)) \rightarrow L(H)$, $f \mapsto f(A)$,

s.t. if $f = p|_{\sigma(A)}$ for a polynomial

$p = a_0 + a_1 x + \dots + a_n x^n$, then $f(A) = p(A)$

($\stackrel{\text{def}}{=} a_0 I + a_1 A + \dots + a_n A^n$).

Moreover, this map is isometric

($\|f(A)\| = \|f\|$) and it is a $\ast$ -homomorphism:

$$(\alpha f + \beta g)(A) = \alpha f(A) + \beta g(A) \quad (\alpha, \beta \in \mathbb{C}),$$

$$(fg)(A) = f(A)g(A), \quad f(A)^* = \overline{f}(A).$$

Proof

For every polynomial p , the operator $p(A)$

is normal, hence $\|p(A)\| = r(p(A))$.

By the spectral mapping theorem we

have $\sigma(p(A)) = p(\sigma(A))$, hence

$$r(p(A)) = \sup \{ |p(\lambda)| : \lambda \in \sigma(A) \}$$

$$= \|p|_{\sigma(A)}\|.$$

Therefore we have a well-defined isometric map

$$\{ \text{polynomial functions on } \sigma(A) \} \rightarrow L(H).$$

$$p|_{\sigma(A)} \mapsto p(A)$$

Since polynomial functions are dense in $C(b(A))$ by the Weierstrass theorem, the above map extends by continuity to an isometry $C(b(A)) \rightarrow L(H)$. ②

This is a $*$ -homomorphism, since all the required properties hold for the polynomial functions. □

Observe a few more properties of the functional calculus:

Lemma

(1) If $f \geq 0$, then $f(A) \geq 0$.

(2) If $f(0) = 0$, ~~f is continuous~~ and $A \in K(H)$, then $f(A) \in K(H)$ (we assumed $\dim H = \infty$, so $0 \in b(A)$). □

Proof

(1) Define $g(x) = f(x)^{1/2}$. Then

$$f(A) = g(A)^2 = g(A)^* g(A) \geq 0.$$

(2) Choose polynomials p_n s.t. $p_n \rightarrow f$ uniformly on $b(A)$.

Then $p_n(0) \rightarrow f(0) = 0$. Hence, replacing p_n by $p_n - p_n(0)$ we may assume that $p_n(0) = 0$. Then $p_n(A) \in K(H)$ and $f(A) = \lim_{n \rightarrow \infty} p_n(A) \in K(H)$. □

Now, given any $T \in L(H)$, we define

$$|T| \stackrel{\text{def}}{=} (T^*T)^{1/2}.$$

Warning

One should not take for granted any familiar properties of the modulus.

For example, even for self-adjoint operators, it is not true that

$$|A+B| \leq |A|+|B|.$$

Proposition

For every positive $A \in L(H)$, the operator $A^{1/2}$ is the unique positive operator with square A .

Proof

Assume $B \geq 0$, $B^2 = A$.

Then $BA = B^3 = AB$, and hence $A^n B = BA^n$ for $\forall n \geq 0$. Since $A^{1/2} = \lim_{n \rightarrow \infty} p_n(A)$ for some polynomials p_n , it follows that $A^{1/2}B = BA^{1/2}$. Then

$$(A^{1/2} - B)(A^{1/2} + B) = (A^{1/2})^2 - B^2 = A - A = 0.$$

Therefore $A^{1/2} = B$ on $\text{Im}(A^{1/2} + B)$.

Note that $\ker(A^{1/2} + B) = (\ker A^{1/2}) \cap (\ker B)$.
Indeed, the inclusion $\supset$ is obvious. For the opposite inclusion, take $x \in \ker(A^{1/2} + B)$.
Then $0 = ((A^{1/2} + B)x, x) = (A^{1/2}x, x) + (Bx, x)$.

hence $(A^{1/2}x, x) = (Bx, x) = 0$. (3)

As $(A^{1/2}x, x) = ((A^{1/2})^2 x, x) = (A^{1/4}x, A^{1/4}x)$,
we get $A^{1/4}x = 0$, hence $A^{1/2}x = 0$. Similarly,
 $Bx = 0$, so $x \in (\ker A^{1/2}) \cap (\ker B)$.

Therefore we also have
 $A^{1/2} = B$ on $\ker(A^{1/2} + B) = (\ker A^{1/2}) \cap (\ker B)$.

Since

$$H = \overline{\operatorname{Im}(A^{1/2} + B)} \oplus (\operatorname{Im}(A^{1/2} + B))^\perp$$

$$= \overline{\operatorname{Im}(A^{1/2} + B)} \oplus \ker(A^{1/2} + B),$$

we see that $A^{1/2} = B$. □

An operator $u \in L(H)$ is called
a partial isometry if there is a
closed subspace $K \subset H$ s.t. $u|_K : K \rightarrow H$
is an isometry and $u|_{K^\perp} = 0$. Then
 K is called the initial space of u and
 $L = \overline{uK}$ is called the final space of u .

In matrix form,

$$u : K \oplus K^\perp \rightarrow L \oplus L^\perp$$

is $\begin{pmatrix} u_{11} & 0 \\ 0 & 0 \end{pmatrix}$, with $u_{11} : K \rightarrow L$ unitary.

Then $u^* : L \oplus L^\perp \rightarrow K \oplus K^\perp$ is $\begin{pmatrix} u_{11}^* & 0 \\ 0 & 0 \end{pmatrix}$

and $u_{11}^* = u_{11}^{-1}$, so $u^*u : H \rightarrow H$ is the

orthogonal projection onto K and $u u^* : H \rightarrow H$ is the orthogonal projection onto $L = uK$.

Theorem (Polar decomposition)

Assume H is a Hilbert space, $T \in L(H)$. Then T can be written uniquely as $T = uA$, where $A \geq 0$ and u is a partial isometry with initial space $\overline{\text{Im } A}$. Moreover, $A = |T|$.

Proof

Existence

$$\text{Put } A = |T| = (T^*T)^{1/2}.$$

Note that $\ker |T| = \ker |T|^2 = \ker T^*T = \ker T$.

(Similarly to an earlier argument: clearly, $\ker T \subset \ker(T^*T)$. If $x \in \ker(T^*T)$, then $0 = (T^*Tx, x) = \|Tx\|^2$, so $x \in \ker T$.)

hence $(\text{Im } |T|)^\perp = \ker |T| = \ker T$, and therefore $\overline{\text{Im } |T|} = (\ker T)^\perp$.

Define $u : \text{Im } |T| \rightarrow H$ by $u(|T|x) = Tx$. This is a well-defined isometric operator, since $\|Tx\|^2 = (T^*Tx, x) = (|T|^2x, x) = \||T|x\|^2$.

Then u extends by continuity to an

isometry $u: \overline{\text{Im } |T|} \rightarrow H$. We then extend it to a partial isometry on H by letting $u=0$ on $(\overline{\text{Im } |T|})^\perp = \ker T$.

Uniqueness

Assume $T = vB$, with $B \geq 0$ and v a partial isometry with initial space $\text{Im } B$. Then v^*v is the projection onto $\overline{\text{Im } B}$, so $v^*vB = B$. Thus, $T^*T = Bv^*vB = B^2$. By uniqueness of the square root we conclude that $B = (T^*T)^{1/2} = |T|$.

Then $v|T|x = Tx = u|T|x$, so $v=u$ on $\text{Im } |T|$, hence on $\overline{\text{Im } |T|}$, and then $v=u$. □

