

We now apply polar decomposition to compact operators.

Assume H is a Hilbert space and $T \in K(H)$.

Let $(s_k(T))_k$ be the nonzero singular values of T . Choose an orthonormal system of unit vectors e_k s.t.

$$|T|e_k = s_k(T)e_k.$$

Then $(e_k)_k$ is an orthonormal basis in $(\ker T)^\perp = (\ker |T|)^\perp$.

Consider the polar decomposition $T = u|T|$.

Let $f_k = ue_k$. Then $(f_k)_k$ is an orthonormal system.

Theorem

With the above notation, T can be written as

$$T = \sum_k s_k(T) (\cdot, e_k) f_k,$$

where the sum is either finite or norm-convergent.

Proof

If T has infinitely many nonzero singular values, then $s_k(T) \rightarrow 0$ as $k \rightarrow \infty$.

This implies that the series $\sum_{k=1}^{\infty} s_k(T) (\cdot, e_k) f_k$ is norm-convergent, since

$$\left\| \sum_{k=n}^m s_k(T) (\cdot, e_k) f_k \right\| = \left\| u \sum_{k=n}^m s_k(T) (\cdot, e_k) e_k \right\|$$

$$= \left\| \sum_{k=1}^m S_k(T) (\cdot, e_k) e_k \right\| = \max_{n \in K \subseteq M} S_n(T) \xrightarrow[n \in M]{n, m \rightarrow \infty} 0.$$

The identity $T = \sum_k S_k(T) (\cdot, e_k) f_k$ holds, because it is obviously true on $\ker T$, while on the vectors e_k the right-hand side gives

$$S_k(T) (e_k, e_k) f_k = S_k(T) f_k = U(T)e_k = Te_k.$$

Sturm-Liouville problem

Consider a finite interval $[a, b] \subset \mathbb{R}$ and the differential operator

$$L = -\frac{d}{dx} \left(p(x) \frac{d}{dx} \right) + q(x)1$$

on $L^2(a, b)$, where $p \in C^1[a, b]$, $p > 0$, $q \in C[a, b]$, $q \geq 0$. More precisely, L is not defined on the whole space $L^2(a, b)$, as its domain of definition we take

$$D(L) \stackrel{\text{def}}{=} \{ u \in C^2[a, b] : u(a) = u(b) = 0 \}$$

(Other boundary conditions are possible, the choice does affect the analysis.)

The Sturm-Liouville problem is to find eigenvalues and eigenvectors of L , that is, solve

$$Lu = \lambda u \quad (u \in D(L), \lambda \in \mathbb{C}).$$

Proposition

The operator L is symmetric, that is,
 $(Lu, v) = (u, Lv)$ for $u, v \in D(L)$.

We also have $(Lu, u) \geq 0$, and $(Lu, u) = 0$ if and only if $u = 0$.

Remark

We don't say that L is self-adjoint, this term is reserved for operators satisfying extra conditions to be discussed in the next course.

Proof

The key point is that the operator $i \frac{d}{dx}$ is symmetric on $\{u \in C^1[a, b]; u(a) = u(b) = 0\}$, since

$$(u', v) + (u, v') = \int_a^b (u'(x) \overline{v(x)} + u(x) \overline{v'(x)}) dx \\ = u \overline{v} \Big|_a^b = 0.$$

We see, in fact, that it suffices to assume $u(a) = u(b) = 0$ or $v(a) = v(b) = 0$.

Using this and the assumption that p and q are real, we have

$$(Lu, v) = -((pu')', v) + (qu, v) \\ = (pu', v') + (u, qv) \\ = (u', pv') + (u, qv) \\ = -(u, (pv')') + (u, qv) = (u, Lv).$$

We see also that

$$\begin{aligned}(Lu, u) &= \int_a^b (pu', u') + (qu, u) \\ &= \int_a^b (p(x)|u'(x)|^2 + q(x)|u(x)|^2) dx.\end{aligned}$$

As $p > 0$ and $q \geq 0$, we see that $(Lu, u) \geq 0$, and if $(Lu, u) = 0$, then $u' = 0$ a.e.. Since $u \in C^2[a, b]$ and $u(a) = u(b) = 0$, this is possible only when $u = 0$. $\square$

Corollary

If λ is an eigenvalue of L , then $\lambda > 0$.

Proof

If $Lu = \lambda u$, $u \neq 0$, then $\lambda \|u\|_2^2 = (Lu, u) > 0$, so $\lambda > 0$. $\square$

To analyze the eigenvalues of L we will show that (an extension of) L^{-1} is compact.

Def

Green's function for L is a continuous function G on $[a, b] \times [a, b]$ s.t.

(i) $G(\cdot, y) \in C^2[a, y]$ and $G(\cdot, y) \in C^2[y, b]$ for all $y \in [a, b]$;

(ii) $G(a, y) = G(b, y) = 0$ for all y ;

(iii) $L_x G(x, y) = 0$ for all y , $a < x < y$, $y < x < b$;

(iv) $\frac{\partial G}{\partial x}(x^+, y) - \frac{\partial G}{\partial x}(x^-, y) = -\frac{1}{p(x)}$, $a < x < b$.

Here L_x means that we apply the operator L in the variable x . (3)

As we will see soon, such a function exists. It can also be shown that it is unique. The explicit formula will justify computations in the next proof:

Proposition

If G is a Green's function and $f \in C[a, b]$, then the function

$$u(x) \stackrel{\text{def}}{=} \int_a^b G(x, y) f(y) dy$$

satisfies $u \in D(L)$ and $Lu = f$.

Proof

It is easy to see that $u \in C[a, b]$,

$$u(a) = u(b) = 0.$$

$$\begin{aligned} u'(x) &= \frac{d}{dx} \left(\int_a^x G(x, y) f(y) dy + \int_x^b G(x, y) f(y) dy \right) \\ &= G(x, x) f(x) + \int_a^x \frac{\partial G}{\partial x}(x, y) f(y) dy \\ &\quad - G(x, x) f(x) + \int_x^b \frac{\partial G}{\partial x}(x, y) f(y) dy \\ &= \int_a^x \frac{\partial G}{\partial x}(x, y) f(y) dy + \int_x^b \frac{\partial G}{\partial x}(x, y) f(y) dy. \end{aligned}$$

Then

$$\begin{aligned} (pu')'(x) &= p(x) \frac{\partial G}{\partial x}(x^+, x) f(x) + \int_a^x \left(\frac{d}{dx} p(x) \frac{d}{dx} \right) (G(x, y) f(y)) dy \\ &\quad - p(x) \frac{\partial G}{\partial x}(x^-, x) f(x) + \int_x^b \left(\frac{d}{dx} p(x) \frac{d}{dx} \right) (G(x, y) f(y)) dy \\ &= -f(x) + \int_a^b \left(\frac{d}{dx} p(x) \frac{d}{dx} \right) (G(x, y) f(y)) dy. \end{aligned}$$

Therefore

$$(Lu)(x) = f(x) + \int_a^b L_x G(x,y) f(y) dy = f(x).$$

In order to show that G exists
consider two solutions of $Lu=0$ satisfying
the initial conditions

$$u_e(a)=0, u_e'(a)=1 \text{ and } u_r(b)=0, u_r'(b)=1.$$

Such solutions u_e, u_r exist by general
results in ODE. Consider the (modified)

Wronskian

$$W(x) = p(x) \det \begin{pmatrix} u_e(x) & u_r(x) \\ u_e'(x) & u_r'(x) \end{pmatrix} \\ = p(x) (u_e(x)u_r'(x) - u_e'(x)u_r(x)).$$

Lemma

The function W is constant and nonzero.

Proof

We have

$$W' = p'(u_e u_r' - u_e' u_r) + \underbrace{p(u_e' u_r' - u_e' u_r')}_{=0} \\ + p(u_e u_r'' - u_e'' u_r).$$

Since $p u_e'' = q u_e - p' u_e'$ and $p u_r'' = q u_r - p' u_r'$,
we have

$$W' = p'(u_e u_r' - u_e' u_r) + p u_e (q u_r - p' u_r') \\ - u_r (q u_e - p' u_e') = 0.$$