Therefore W is constant.

Assume $W=0$. Then $\det\begin{pmatrix} u_e(a) & u_0(a) \\ u_e'(a) & u_0'(a) \end{pmatrix}=0$, hence the function $u(x) \overset{\text{def}}{=} u_0(x) - u_0'(a)u_e(x)$ satisfies $Lu=0$, $u(a)=0$, $u'(a)=0$. But the only such function in $C^2[a,b]$ is 0, so $u_0 = u_0'(a)u_e$. It follows that u_0 satisfies $Lu_0=0$, $u_0(a)=u_0(b)=0$. As we showed, the only such function is 0 (as $u_0 \in D(L)$, $(Lu_0, u_0)=0$). This is a contradiction. ∎

We now define

$$G(x,y) \overset{\text{def}}{=} -\frac{1}{W}\begin{cases} u_e(x)u_0(y), & \text{if } x \le y, \\ u_e(y)u_0(x), & \text{if } x \ge y. \end{cases}$$

Lemma

G is a Green's function for L.

Proof

We will only check condition (iii) in the definition, the other conditions are trivially satisfied.

$$\frac{\partial G}{\partial x}(x^+,x) = -\frac{1}{W}u_e(x)u_0'(x),$$

$$\frac{\partial G}{\partial x}(x^-,x) = -\frac{1}{W}u_e'(x)u_0(x), \text{ so}$$

$$\frac{\partial G}{\partial x}(x^+,x) - \frac{\partial G}{\partial x}(x^-,x) = -\frac{u_e(x)u_0'(x) - u_e'(x)u_0(x)}{W} = -\frac{1}{p(x)}.$$

∎

To summarize, L is a bijection

$$D(L) = \{u \in C^2[a,b] : u(a) = u(b) = 0\} \to C[a,b]$$

and the inverse is given by

$$(L^{-1} f)(x) = \int_a^b G(x,y) f(y)\, dy.$$

This operator extends by continuity to $L^2(a,b)$. Namely, given a Borel function $K : [a,b] \times [a,b] \to \mathbb{C}$, the integral operator with kernel K is defined, for suitable f, by

$$(T_K f)(x) \overset{\text{def}}{=} \int_a^b K(x,y) f(y)\, dy$$

Proposition

If $K \in C([a,b] \times [a,b])$, then T_K is well-defined compact operator on $L^2(a,b)$, and $T_K^* = T_{K^*}$, where $K^*(x,y) \overset{\text{def}}{=} \overline{K(y,x)}$.

(we will later see that the same is true for a larger class of kernels.)

Proof

Since $L^2(a,b) \subset L^1(a,b)$, $T_K f$ is a well-defined measurable function for all $f \in L^2(a,b)$.

$$|(T_K f)(x)| \underset{CS}{\leq} \left(\int_a^b |K(x,y)|^2 dy \right)^{1/2} \left(\int_a^b |f(y)|^2 dy \right)^{1/2}$$

Thus, $\|T_K f\|_\infty \leq M \|f\|_2$,

where $M = \max_x \left(\int_a^b |K(x,y)|^2 dy \right)^{1/2}$.

Since K is continuous, $\forall \varepsilon > 0$ we can find $\delta > 0$ s.t.

$$|K(x_1,y) - K(x_2,y)| < \varepsilon \quad \text{if} \quad |x_1 - x_2| < \delta.$$

Then, similarly to the above inequality,

$$|(T_K f)(x_1) - (T_K f)(x_2)| \le \left(\int_a^b |K(x_1,y) - K(x_2,y)|^2 dy \right)^{1/2} \|f\|_2$$

$$\le \varepsilon (b-a)^{1/2} \|f\|_2.$$

By Arzelà-Ascoli it follows that

$$\{ T_K f : f \in L^2(a,b), \|f\|_2 \le 1 \} \subset C[a,b]$$

is precompact. In other words, T_K is compact as an operator $L^2(a,b) \to C[a,b]$. Since the embedding map $C[a,b] \to L^2(a,b)$ is bounded, T_K is also compact as an operator on $L^2(a,b)$.

Finally, if $f, g \in L^2(a,b)$, then by Fubini we have

$$(T_K f, g) = \int_a^b \left(\int_a^b K(x,y) f(y) dy \right) \overline{g(x)} dx$$

$$= \int_a^b \left(\int_a^b \overline{K(x,y)}\, g(x) dx \right) f(y) dy$$

$$= (f, T_{K^*} g).$$

Returning to L, we see that $L^{-1} = T_G$ on $C[a,b]$ and $T_G^* = T_G$, since G is real-valued and $G(x,y) = G(y,x)$.

Theorem

Consider $Lu = -(pu')' + qu$, with $p \in C^1[a,b]$, $p > 0$, $q \in C[a,b]$, $q \geq 0$, and

$$D(L) = \{u \in C^2[a,b] : u(a) = u(b) = 0\}.$$

Then

(1) every eigenvalue of L is strictly positive and simple;

(2) the eigenvalues form an increasing sequence $\lambda_1 < \lambda_2 < \dots$ and $\lambda_n \xrightarrow[n\to\infty]{} \infty$;

(3) if u_n are the corresponding normalized eigenvectors, so that $Lu_n = \lambda_n u_n$ and $\|u_n\|_2 = 1$, then $(u_n)_{n=1}^{\infty}$ is an orthonormal basis in $L^2(a,b)$.

Proof

(1) We already know that every eigenvalue λ is strictly positive.

If $u, v \in D(L)$ are nonzero and satisfy $Lu = \lambda u$, $Lv = \lambda v$, then by swapping u and v if necessary we have $u'(a) = c\,v'(a)$ for some $c \in \mathbb{R}$. Then $w \stackrel{\text{def}}{=} u - cv \in C^2[a,b]$ satisfies $Lw = 0$, $w(a) = w'(a) = 0$, hence $w = 0$, so $u = cv$.

Thus, the multiplicity of λ is 1.

(2) Consider the operator T_G on $L^2(a,b)$, where G is Green's function. As we have shown, T_G is self-adjoint and compact. We have also shown that $\operatorname{Im} T_G \subset C[a,b]$ and $LT_G = 1$ on $C[a,b]$. Since $\ker L = 0$, it follows then that $T_G(C[a,b]) = D(L)$ and $T_G L = 1$ on $D(L)$.

We claim that $\ker T_G = 0$. Indeed, if $T_G f = 0$, then for all $g \in C[a,b]$ we have $0 = (T_G f, g) = (f, T_G g)$. Thus $f \in \left(T_G(C[a,b])\right)^\perp = D(L)^\perp = 0$, since already $C_c^\infty(a,b) \subset D(L)$ is dense in $L^2(a,b)$.

Next, we claim that if $f \neq 0$ is an eigenvector of T_G, so $T_G f = \mu f$ for some $\mu \neq 0$, then $f \in D(L)$ and $Lf = \frac{1}{\mu} f$. Indeed, since $\operatorname{Im} T_G \subset C[a,b]$, we have $f \in C[a,b]$. Then $f = \frac{1}{\mu} T_G f \in D(L)$ and $Lf = \frac{1}{\mu} LT_G f = \frac{1}{\mu} f$. Similarly, if $f \in D(L)$, $f \neq 0$, is an eigenvector of L with eigenvalue λ, then $T_G f = \frac{1}{\lambda} f$. Then (2) and (3) follow from the spectral theorem for T_G. ∎

Example

Consider the laplacian $\Delta = -\frac{d^2}{dx^2}$ on $(0,1)$.

The solutions of

$$\begin{cases} -u'' = \lambda u & (u \in C^2[0,1]) \\ u(0) = u(1) = 0, \end{cases}$$

up to normalization, are

$$u_n = \sqrt{2}\sin n\pi x, \quad \lambda_n = \pi^2 n^2, \quad n \geq 1.$$

In particular, we prove (again) that $(\sqrt{2}\sin n\pi x)_{n=1}^{\infty}$ is an orthonormal basis in $L^2(0,1)$.

Remark

The assumption $q \geq 0$ was not important, we can always reduce to this case by adding $m1$ to L, with $m > 0$ large enough.

Fact (Weyl asymptotics)

Let $\lambda_1 < \lambda_2 < \dots$ be the eigenvalues of a Sturm-Liouville operator (with $p \in C^1[a,b]$, $p > 0$, $q \in C[a,b]$ real). Then

$$\lambda_n = \left(\frac{\pi n}{\ell}\right)^2 + O(n),$$

where $\ell = \int_a^b \frac{dx}{p(x)^{1/2}}$.