# Trace-class and Hilbert-Schmidt operators

**Proposition**

Assume $\mathcal{H}$ is a Hilbert space and $(e_i)_{i\in I}$ is an orthonormal basis in $\mathcal{H}$. For $T\in L(\mathcal{H})$, $T\geq 0$, define

$$\mathrm{Tr}\,T \overset{\text{def}}{=} \sum_i (Te_i, e_i) \in [0,+\infty].$$

Then $\mathrm{Tr}\,T$ is independent of the choice of $(e_i)_i$.

**Proof**

Assume $(f_j)_j$ is another orthonormal basis. Using the Parseval identity twice we get

$$\begin{aligned}\sum_i (Te_i,e_i) &= \sum_i (T^{1/2}e_i, T^{1/2}e_i) = \sum_i \|T^{1/2}e_i\|^2 \\ &= \sum_i \sum_j |(T^{1/2}e_i, f_j)|^2 \\ &= \sum_j \sum_i |(T^{1/2}f_j, e_i)|^2 \\ &= \sum_j \|T^{1/2}f_j\|^2 = \sum_j (Tf_j, f_j).\end{aligned}$$

∎

**Def.**

An operator $T\in L(\mathcal{H})$ is said to be of trace class if $\mathrm{Tr}\,|T| < \infty$. We denote by $S_1(\mathcal{H})$ the set of trace-class operators.

Since the function $T\mapsto |T|$ is badly behaved, it is not obvious that $S_1(\mathcal{H})$ is a subspace of $L(\mathcal{H})$.

Example

Consider the self-adjoint operators

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}.$$

$$|A| = A, \quad |B| = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},$$

$$|A+B| = \left| \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right| = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix},$$

since $\frac{1}{\sqrt{2}}(A+B)$ is unitary. Thus,

$$|A+B| = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}, \quad |A|+|B| = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}, \text{ and}$$

$$(|A+B| e_1, e_1) = \sqrt{2} > 1 = ((|A|+|B|) e_1, e_1), \text{ so}$$

$$|A+B| \not\leq |A|+|B|.$$

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Nevertheless, we will see soon that $S_1(H)$ is a vector space with norm

$$\|T\|_1 \stackrel{\text{def}}{=} \mathrm{Tr}\,|T|, \text{ called the } \underline{\text{trace norm}}$$

It is convenient to first analyze a larger class of operators.

Def

An operator $T \in L(H)$ is called Hilbert-Schmidt if $\mathrm{Tr}(T^*T) < \infty$. The set of such operators is denoted by $S_2(H)$ or $HS(H)$. We put

$$\|T\|_2 = \|T\|_{HS} \stackrel{\text{def}}{=} \left[\mathrm{Tr}(T^*T)\right]^{1/2}.$$

More generally, given Hilbert spaces $H$ and $K$, an operator $T \in L(H,K)$ is

called Hilbert-Schmidt if $\mathrm{Tr}(T^*T) < \infty$.
Since any $T \in L(H,K)$ can be viewed as an operator $\begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}$ on $H \oplus K$, the general case reduces to the case of one space.

**Theorem**

Assume $H$ is a Hilbert space. Then $S_2(H)$ is a Hilbert space with norm $\|\ \|_2$. Moreover,

(1) $F(H) \subset S_2(H) \subset K(H)$;

(2) if $T \in S_2(H)$, then $T^* \in S_2(H)$ and $\|T^*\|_2 = \|T\|_2$;

(3) if $T \in S_2(H)$, $S \in L(H)$, then
$ST, TS \in S_2(H)$ and $\|ST\|_2 \le \|S\| \cdot \|T\|_2$, $\|TS\|_2 \le \|T\|_2 \|S\|$.

Properties (2) and (3) mean that $S_2(H)$ is a *-ideal* in $L(H)$.

**Proof**

Choose an orthonormal basis $(e_i)_{i \in I}$ in $H$.
Then $\mathrm{Tr}(T^*T) = \sum_i \|Te_i\|^2$. By the triangle inequality for $\ell^2(I)$ we then see that $S_2(H)$ is a vector space with norm $\|\ \|_2$, since

$$\left( \sum_i \|(S+T)e_i\|^2 \right)^{1/2} \le \left( \sum_i (\|Se_i\| + \|Te_i\|)^2 \right)^{1/2}$$
$$\le \left( \sum_i \|Se_i\|^2 \right)^{1/2} + \left( \sum_i \|Te_i\|^2 \right)^{1/2}.$$

Since $|(Se_i, Te_i)| \le \|Se_i\| \cdot \|Te_i\|$, we see

also that the norm $\|\ \|_2$ is defined by the scalar product

$$(S,T)_{HS} \overset{\text{def}}{=} \sum_i (Se_i, Te_i).$$

Since a scalar product is determined by the corresponding norm, we see in particular that the above expression is independent of the choice of $(e_i)_i$.

To show that $S_2(H)$ is complete, observe first that $\|T\| \le \|T\|_2$. Indeed, for any unit vector $x$ we can find an orthonormal basis $(f_j)_j$ such that $x = f_{j_0}$ for some $j_0$. Then $\|T\|_2 = \left(\sum_j \|Tf_j\|^2\right)^{1/2} \ge \|Tx\|$, which proves the claim.

Assume now that $(T_n)_n$ is a Cauchy sequence in $S_2(H)$. Then $(T_n)_n$ is a Cauchy sequence in $(L(H), \|\ \|)$, so there is $T \in L(H)$ s.t. $\|T - T_n\| \underset{n\to\infty}{\to} 0$. We claim that $T \in S_2(H)$ and $\|T - T_n\|_2 \underset{n\to\infty}{\to} 0$.

Fix $\varepsilon > 0$ and choose $n_0 \ge 1$ s.t. $\|T_n - T_m\|_2 < \varepsilon$ for all $n, m \ge n_0$. For any finite set $F \subset I$ we then have

$$\left(\sum_{i\in F} \|T_n e_i - T_m e_i\|^2\right)^{1/2} < \varepsilon \quad \text{for } n, m \ge n_0.$$

letting $m \to \infty$ we then get

that $\left(\sum_{i\in E} \|T_n e_i - T e_i\|^2\right)^{1/2} \le \varepsilon$ for $n \ge n_0$.
It follows that $\|T_n - T\|_2 \le \varepsilon$. Thus,
$T = T_n - (T_n - T) \in S_2(H)$, and as $\varepsilon > 0$ was
arbitrary, we see also that $\|T_n - T\|_2 \xrightarrow[n\to\infty]{} 0$.

It remains to check (1)-(3).

(1) $F(H)$ is spanned by the operators
$S = (\cdot, x)y$. We have $S^* = (\cdot, y)x$,
$S^*S = (\cdot, x)\|y\|^2 x$,
$\mathrm{Tr}(S^*S) = \sum_i \|y\|^2 |(x, e_i)|^2 = \|x\|^2 \cdot \|y\|^2$.
Therefore $S \in S_2(H)$ and $\|S\|_2 = \|x\| \|y\|$.

To see that $S_2(H) \subset K(H)$, take any
$T \in S_2(H)$. For $\varepsilon > 0$, choose a finite set
$F \subset I$ s.t. $\sum_{i\notin F} \|T e_i\|^2 < \varepsilon^2$. Let $p$ be the
orthogonal projection onto $\mathrm{span}\{e_i : i \in F\}$.
Then $Tp \in F(H) \subset S_2(H)$ and
$$\|T - Tp\|_2^2 = \sum_i \|Te_i - Tpe_i\|^2 = \sum_{i\notin F} \|Te_i\|^2 < \varepsilon^2,$$
hence $\|T - Tp\| \le \|T - Tp\|_2 < \varepsilon$. As $\varepsilon > 0$
was arbitrary, we conclude that
$T \in \overline{F(H)}^{\|\ \|} = K(H)$.

(2) Take $T \in S_2(H)$.

We have

$$\|T\|_2^2 = \sum_i \|Te_i\|^2 = \sum_{i,j} |(Te_i, e_j)|^2.$$

As $|(Te_i, e_j)| = |(T^* e_j, e_i)|$, it follows that $T^* \in S_2(H)$ and $\|T^*\|_2 = \|T\|_2$.

(3) Take $T \in S_2(H)$ and $S \in L(H)$. We have

$S^*S \leq \|S\|^2 I$, since

$$(S^*Sx, x) = \|Sx\|^2 \leq \|S\|^2 \|x\|^2 = \|S\|^2 (x, x).$$

Hence $T^* S^* S T \leq \|S\|^2 T^* T$, since

$$(T^* S^* S T x, x) = (S^* S Tx, Tx) \leq \|S\|^2 (Tx, Tx) = \|S\|^2 (T^* T x, x).$$

It follows that

$$\operatorname{Tr}(T^* S^* S T) \leq \|S\|^2 \operatorname{Tr}(T^* T),$$

that is, $ST \in S_2(H)$ and $\|ST\|_2 \leq \|S\| \cdot \|T\|_2$.

Using this together with (2) we also get

$$\|TS\|_2 = \|S^* T^*\|_2 \leq \|S^*\| \cdot \|T^*\|_2 = \|S\| \cdot \|T\|_2.$$

We now return to trace-class operators. ∎