Proposition

Assume H is a Hilbert space. Then

$$S_1(H) = \{ST : S, T \in S_2(H)\}$$

$$= \text{span}\{ST : S, T \in S_2(H)\}.$$

Moreover, given $T \in S_1(H)$, we have $\|T\|_1 = \inf \sum_{k=1}^{n} \|S_k\|_2 \|T_k\|_2$, where the infimum is taken over all decompositions $T = \sum_{k=1}^{n} S_k T_k$ with $S_k, T_k \in S_2(H)$.

Proof

If $T \in S_1(H)$, then by definition $|T|^{1/2} \in S_2(H)$. Consider the polar decomposition $T = u|T|$. Then $u|T|^{1/2}, |T|^{1/2} \in S_2(H)$ and $T = u|T|^{1/2} \cdot |T|^{1/2}$. Moreover, $\|u|T|^{1/2}\|_2 \cdot \||T|^{1/2}\|_2 = \||T|^{1/2}\|_2^2$ (see remark below)

$= \operatorname{Tr}|T| = \|T\|_1$.

To finish the proof of the proposition it now suffices to show that if $T \in L(H)$, $T = \sum_{k=1}^{n} S_k T_k$ for some $S_k, T_k \in S_2(H)$, then $T \in S_1(H)$ and $\|T\|_1 \leq \sum_k \|S_k\|_2 \|T_k\|_2$.

Consider again the polar decomposition $T = u|T|$. Then $|T| = u^* T$ and for any orthonormal basis $(e_i)_i$ in H we have

$$\begin{aligned}
\mathrm{Tr}|T| &= \sum_i (u^* T^* e_i, e_i) = \sum_{k,i} (u^* S_k T_k e_i, e_i) \\
&= \sum_{k,i} (T_k e_i, S_k^* u e_i) \\
&\le \sum_{k,i} \|T_k e_i\| \cdot \|S_k^* u e_i\| \\
&\le \sum_k \Big(\sum_i \|T_k e_i\|^2 \Big)^{1/2} \Big(\sum_i \|S_k^* u e_i\|^2 \Big)^{1/2} \\
&= \sum_k \|T_k\|_2 \|S_k^* u\|_2 \\
&\le \sum_k \|T_k\|_2 \|S_k^*\|_2 = \sum_k \|S_k\|_2 \|T_k\|_2 .
\end{aligned}$$

∎

Remark

If $T = u|T|$ is the polar decomposition of $T \in L(H)$, then for any function $f \in C(\sigma(|T|))$ with $f(0) = 0$ (if $0 \in \sigma(|T|)$), we have $\mathrm{Im}\, f(|T|) \subset \overline{\mathrm{Im}|T|}$, since $f(|T|)$ can be approximated by polynomials of $|T|$ with zero constant term. Hence $\|u f(|T|) x\| = \|f(|T|) x\|$ for any $x \in H$.

Theorem

Assume H is a Hilbert space. Then $(S_1(H), \|\cdot\|_1)$ is a normed space and

(1) $F(H) \subset S_1(H) \subset S_2(H)$ $(\subset K(H))$;

(2) if $T \in S_1(H)$, then $T^* \in S_1(H)$ and $\|T^*\|_1 = \|T\|_1$;

(3) if $T \in S_1(H)$, $S \in L(H)$, then $ST, TS \in S_1(H)$ and $\|ST\| \le \|S\| \cdot \|T\|_1$, $\|TS\| \le \|T\|_1 \cdot \|S\|$.

(We will see later that $S_1(H)$ is a Banach space.)

Proof

From the previous proposition we see that $S_1(H)$ is a vector space and $\|\ \|_1$ satisfies the triangle inequality.

Note also that since $\|T\| = \||T|\| = \sup_{\|x\|\leq 1} (|T|x, x)$, we have $\|T\| \leq \|T\|_1$. Since we also have $|\lambda T| = |\lambda| \cdot |T|$ $(\lambda \in \mathbb{C})$ and hence $\|\lambda T\|_1 = |\lambda| \cdot \|T\|_1$, it follows that $\|\ \|_1$ is a norm.

(1) By the previous proposition, $S_1(H) = \{ST : S, T \in S_2(H)\} \subset S_2(H)$, since $S_2(H) \subset L(H)$ is an ideal.

(Moreover, $\|T\|_2 \leq \|T\|_1$, since $\|T\|_2 = \left(\sum_k s_k(T)^2\right)^{1/2}$, $\|T\|_1 = \sum_k s_k(T)$, where $(s_k(T))_k$ are the nonzero singular values of T.)

$F(H) \subset S_1(H)$, for example, because $S_2(H) \supset F(H)$ and every $S \in F(H)$ can be written as PS for a finite-rank projection P.

(2), (3) follow from the previous proposition and the corresponding properties of $S_2(H)$.

For example, for (2), given $T \in S_1(H)$, write $T = T_1 T_2$ with $\|T_1\|_2 \|T_2\|_2 = \|T\|_1$. Then

~~$\|T_2^*\|_2 \|T_1^*\|_2 = \|T_2\|_2 \|T_1\|_2 \leq \|T\|_1$~~, hence

$T^* = T_2^* T_1^* \in S_1(H)$ and $\|T^*\|_1 \leq \|T\|_1$.
By symmetry, $\|T\|_1 \leq \|T^*\|_1$, hence
$\|T^*\|_1 = \|T\|_1$.

Remark (2) follows also from $|T^*| = U|T|U^*$.

We can now extend the trace to non positive operators.

Theorem
Assume H is a Hilbert space. Then Tr extends uniquely from $\{T \geq 0 : \mathrm{Tr}\, T < \infty\}$ to a linear functional on $S_1(H)$. We have:

(1) for any $T \in S_1(H)$ and any orthonormal basis $(e_i)_i$, we have $\mathrm{Tr}\, T = \sum_i (Te_i, e_i)$, where the series is absolutely convergent;

(2) $|\mathrm{Tr}\, T| \leq \mathrm{Tr}\, |T| = \|T\|_1$ $\quad (T \in S_1(H))$;

(3) $\mathrm{Tr}(ST) = \mathrm{Tr}(TS)$ whenever $S, T \in S_2(H)$ or $S \in S_1(H)$ and $T \in L(H)$.

Proof
We need to show that $S_1(H)$ is spanned by positive operators. Then the existence and uniqueness of an extension follows by the same argument as one uses to define the Lebesgue integral of complex functions from positive ones. (The existence will also follow from the proof of (1).)
$S_1(H)$ is spanned by self-adjoint operators, since if $T \in S_1(H)$, then

both $\frac{T+T^*}{2}$ and $\frac{T-T^*}{2i}$ are self-adjoint and lie in $S_1(H)$.

If $T=T^* \in S_1(H)$, then $T_+ \overset{\text{def}}{=} \frac{|T|+T}{2}$ and $T_- = \frac{|T|-T}{2}$ are both positive, of trace class and $T = T_+ - T_-$.

(1) Given $T \in S_1(H)$, write $T = T_1 T_2$ for $T_1, T_2 \in S_2(H)$. Then

$$|(Te_i, e_i)| \le \|T_1^* e_i\| \cdot \|T_2 e_i\|.$$

As $\sum_i \|T_1^* e_i\|^2 < \infty$ and $\sum_i \|T_2 e_i\|^2 < \infty$, it follows that the series $\sum_i (Te_i, e_i)$ is absolutely convergent.

We have $\operatorname{Tr} T = \sum_i (Te_i, e_i)$, since the equality holds on positive T in $S_1(H)$, and such T span the entire space $S_1(H)$.

(2) Write again $T \in S_1(H)$ as $T_1 T_2$ with $T_1, T_2 \in S_2(H)$ and $\|T\|_1 = \|T_1\|_2 \|T_2\|_2$. Then

$$|\operatorname{Tr} T| \le \sum_i |(Te_i, e_i)| \le \sum_i \|T_1^* e_i\| \cdot \|T_2 e_i\|$$
$$\le \left(\sum_i \|T_1^* e_i\|^2\right)^{1/2} \left(\sum_i \|T_2 e_i\|^2\right)^{1/2}$$
$$= \|T_1^*\|_2 \|T_2\|_2 = \|T_1\|_2 \|T_2\|_2 = \|T\|_1.$$

(3) We already know that $\operatorname{Tr}(T^*T) = \operatorname{Tr}(TT^*)$ for all $T \in S_2(H)$. By the polarization

identity we then get $\mathrm{Tr}(S^*T) = \mathrm{Tr}(TS^*)$ for all $S, T \in S_2(H)$. Equivalently, $\mathrm{Tr}(ST) = \mathrm{Tr}(TS)$ for all $S, T \in S_2(H)$.

Now take $S \in S_1(H)$, $T \in L(H)$. Write $S = S_1 S_2$, with $S_1, S_2 \in S_2(H)$. Then

$$\mathrm{Tr}(ST) = \mathrm{Tr}(S_1 S_2 T) = \mathrm{Tr}(S_2 T S_1) = \mathrm{Tr}(T S_1 S_2)$$
$$= \mathrm{Tr}(TS).$$

Remark For $S, T \in S_2(H)$, we have $(S,T)_{HS} = \mathrm{Tr}(T^* S)$.

Recall that $c_0^* \simeq \ell_1$, $\ell_1^* \simeq \ell_\infty$. We will next prove an operator analogue of this:

$$K(H)^* \simeq S_1(H), \quad S_1(H)^* \simeq L(H).$$

Theorem Assume H is a Hilbert space. Then the map $S_1(H) \to K(H)^*$, $T \mapsto \mathrm{Tr}(\cdot T)$, is an isometric isomorphism.

Proof For $T \in S_1(H)$, consider the linear functional $\varphi_T \overset{\text{def}}{=} \mathrm{Tr}(\cdot T)$ on $K(H)$. For $S \in K(H)$,

$$|\varphi_T(S)| = |\mathrm{Tr}(ST)| \leq \|ST\|_1 \leq \|S\| \cdot \|T\|_1,$$

hence $\|\varphi_T\| \leq \|T\|_1$. To prove the equality, consider the polar decomposition $T = U|T|$ and choose an orthonormal basis $(e_i)_{i \in I}$ in $\overline{\mathrm{Im}\,|T|} = (\ker T)^\perp$. For a finite set $F \subset I$, let P_F be the orthogonal projection onto $\mathrm{span}\{e_i : i \in F\}$. Then $\varphi_T(P_F U^*) = \mathrm{Tr}(P_F |T|) = \sum_{i \in F} (|T| e_i, e_i)$. As $\|P_F U^*\| \leq 1$, letting $F \nearrow I$ we get $\|\varphi_T\| \geq \mathrm{Tr}\,|T| = \|T\|_1$.