Therefore the map $S_1(H) \to K(H)^*$, $T \mapsto \varphi_T$, is isometric. To finish the proof it remains to show that it is surjective.

Take $\varphi \in K(H)^*$. Observe that if $\varphi = \varphi_T$, then for each operator $\theta_{x,y} \overset{\text{def}}{=} (\cdot, y)x$ we have

$$\begin{aligned}\varphi(\theta_{x,y}) &= \mathrm{Tr}(\theta_{x,y}T) = \mathrm{Tr}(T\theta_{x,y}) \\ &= \mathrm{Tr}(\theta_{Tx,y}) = \sum_i (e_i, y)(Tx, e_i) \\ &= (Tx, \sum_i (y, e_i)e_i) = (Tx, y).\end{aligned}$$

Motivated by this, we define a sesquilinear form $B: H \times H \to \mathbb{C}$ by $B(x,y) \overset{\text{def}}{=} \varphi(\theta_{x,y})$. We have

$$|B(x,y)| = |\varphi(\theta_{x,y})| \le \|\varphi\| \cdot \|\theta_{x,y}\| = \|\varphi\| \cdot \|x\| \cdot \|y\|.$$

It follows that there exists a unique $T \in L(H)$ s.t. $B(x,y) = (Tx, y)$. The above computation shows that $\varphi(\theta_{x,y}) = \mathrm{Tr}(\theta_{x,y}T)$ for all $x, y \in H$, hence $\varphi(S) = \mathrm{Tr}(ST)$ for all $S \in F(H)$.

If we can prove that $T \in S_1(H)$, then $\varphi = \varphi_T$ on $F(H)$ and hence $\varphi = \varphi_T$ by density of $F(H)$ in $K(H)$, and we are done. But this is essentially done at the beginning of the proof; using

the same (e_i); and FCI, we have

$$\varphi(P_F U^*) = \mathrm{Tr}(P_F U^* T) = \sum_{i \in F} (|T| e_i, e_i),$$

so that $\sum_i (|T| e_i, e_i) \leq \|\varphi\|$ and therefore $T \in S_1(H)$. ∎

Corollary

The space $(S_1(H), \|\ \|_1)$ is complete.

Theorem

Assume H is a Hilbert space. Then the map $L(H) \to S_1(H)^*$, $T \mapsto \mathrm{Tr}(\cdot\, T)$, is an isometric isomorphism.

Proof

For $T \in L(H)$, consider the linear functional $\varphi_T \overset{\text{def}}{=} \mathrm{Tr}(\cdot\, T)$ on $S_1(H)$. For $S \in S_1(H)$,

$$|\varphi_T(S)| = |\mathrm{Tr}(ST)| \leq \|ST\|_1 \leq \|S\|_1 \|T\|,$$

hence $\|\varphi_T\| \leq \|T\|$. On the other hand,

$$|\varphi_T(\theta_{x,y})| = |\mathrm{Tr}(\theta_{x,y} T)| = |(Tx, y)|,$$

and since $\|\theta_{x,y}\|_1 = \|x\| \cdot \|y\|$, we set (see below)

$$\|\varphi_T\| \geq \sup_{\|x\|, \|y\| \leq 1} |\varphi_T(\theta_{x,y})| = \sup_{\|x\|, \|y\| \leq 1} |(Tx, y)| = \|T\|$$

Thus, $\|\varphi_T\| = \|T\|$.

(If $y \neq 0$, then $\theta_{x,y}^* \theta_{x,y} = \theta_{y,x} \theta_{x,y} = \|x\|^2 \theta_{y,y}$ and $|\theta_{x,y}| = \|x\| \cdot \|y\| \, \theta_{\frac{y}{\|y\|}, \frac{y}{\|y\|}}$, so $\mathrm{Tr}\,|\theta_{x,y}| = \|x\| \cdot \|y\|$.)

Therefore the map $L(H) \to S_1(H)^*$, $T \mapsto \psi_T$, is isometric. It remains to check that it is surjective.

Take $\psi \in S_1(H)^*$. The same arguments as in the proof of the previous theorem show that there exists $T \in L(H)$ s.t. $\psi(\theta_{x,y}) = \mathrm{Tr}(\theta_{x,y} T)$ $\forall x, y \in H$. Then $\psi = \psi_T$ on $F(H) \subset S_1(H)$, and to conclude that $\psi = \psi_T$ it suffices to show that $F(H)$ is dense in $S_1(H)$ in the norm $\| \|_1$.

Let $S \in S_1(H)$. Write $S = S_1 S_2$, with $S_1, S_2 \in S_2(H)$ and $\|S\|_1 = \|S_1\|_2 \|S_2\|_2$. As we showed in the proof of the inclusion $S_2(H) \subset K(H)$, $\forall \varepsilon > 0$ there is a finite-rank projection p s.t. $\|S_2 - S_2 p\|_2 < \varepsilon$. Then $\|S - Sp\|_1 = \|S_1(S_2 - S_2 p)\|_1 \leq \|S_1\|_2 \|S_2 - S_2 p\|_2$ $\leq \varepsilon \|S_1\|_2$. As $\varepsilon > 0$ was arbitrary, we conclude that $S \in \overline{F(H)}^{\| \|_1}$. ∎

Remark

More generally, for $1 \leq p < \infty$, the Schatten p-class is defined by

$$S_p(H) \overset{\text{def}}{=} \{T \in L(H) : \mathrm{Tr}(|T|^p) < \infty\}.$$

The Schatten p-norm is $\|T\|_p \overset{\text{def}}{=} \mathrm{Tr}(|T|^p)^{1/p}$.

It can be shown that $S_p(H)$ is a $*$-ideal in $L(H)$, and if $1<p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$, then $ST \in S_1(H)$ for $S \in S_p(H)$, $T \in S_q(H)$, and the map $S_q(H) \to S_p(H)^*$, $T \mapsto \tau_0(\cdot T)$, is an isometric isomorphism.

In the proof of the previous theorem we showed that $E(H)$ is dense in $S_1(H)$ and $S_2(H)$. This can be made more explicit.

Def
A sequence $(T_n)_{n=1}^{\infty}$ in $L(H)$ is said to converge to $T \in L(H)$ in the strong operator topology, or simply strongly, if
$\|Tx - T_n x\| \xrightarrow[n\to\infty]{} 0 \quad \forall x \in H$.

Example
Assume H is a separable Hilbert space with an orthonormal basis $(e_n)_{n=1}^{\infty}$. Let P_n be the orthogonal projection onto $\operatorname{span}\{e_k : 1 \le k \le n\}$. Then $P_n \xrightarrow[n\to\infty]{} 1$ strongly.

Theorem
Assume H is a Hilbert space and $(S_n)_{n=1}^{\infty}$ is a sequence of operators converging strongly to 1. Then, for any $T \in S_2(H)$,

we have $TS_n^* \to T$, $S_nT \to T$, $S_nTS_n^* \to T$ in $(S_2(H), \|\ \|_2)$.

Proof

By the uniform boundedness principle we have $C \overset{\text{def}}{=} \sup_n \|S_n\| < \infty$.

Then $\|TS_n^*\|_2 \leq \|T\|_2 \|S_n^*\| \leq C\|T\|_2$. It follows that to prove that $\|TS_n^* - T\|_2 \to 0$ it suffices to check this on operators T spanning a dense subspace of $S_2(H)$. Thus, it is enough to consider $T = \theta_{x,y} = (\cdot, y)x$.

We have $\theta_{x,y}S_n^* = (S_n^*\cdot, y)x = (\cdot, S_n y)x = \theta_{x,S_n y}$.

Then

$$\|\theta_{x,y} - \theta_{x,y}S_n^*\|_2 = \|\theta_{x,y} - \theta_{x,S_n y}\|_2$$

$$= \|\theta_{x,y-S_n y}\|_2 = \|x\| \cdot \|y - S_n y\| \underset{n\to\infty}{\to} 0.$$

Similarly, using that $S_n\theta_{x,y} = \theta_{S_n x, y}$ one shows that $\|S_nT - T\|_2 \underset{n\to\infty}{\to} 0 \quad \forall\, T \in S_2(H)$.

Finally,

$$\|S_nTS_n^* - T\|_2 \leq \|S_nTS_n^* - S_nT\|_2 + \|S_nT - T\|_2$$

$$\leq \|S_n\| \cdot \|TS_n^* - T\|_2 + \|S_nT - T\|_2$$

$$\leq C\|TS_n^* - T\|_2 + \|S_nT - T\|_2 \to 0.$$

□

Corollary

If $T \in S_1(H)$, then $TS_n^* \to T$, $S_nT \to T$, $S_nTS_n^* \to T$ in $(S_1(H), \|\ \|_1)$.

Proof

The convergences $TS_n^* \to T$, $S_n T \to T$ are proved by decomposing $T = T_1 T_2$, with $T_1, T_2 \in S_2(H)$, and applying the theorem.

Then the convergence $S_n T S_n^* \to T$ is deduced from this ~~example~~ as in the proof of the theorem. ∎

Corollary

If $T \in S_1(H)$, then $\operatorname{Tr}(S_n T S_n^*) \to \operatorname{Tr} T$.

Example

Assume $(P_n)_{n=1}^{\infty}$ is a sequence of finite-rank orthogonal projections s.t. $P_n \to 1$ strongly. Choose an orthonormal basis $(e_{ni})_{i=1}^{m_n}$ in $P_n H$. Then we conclude that if $T \in S_1(H)$, then

$$\operatorname{Tr} T = \lim_{n \to \infty} \sum_{i=1}^{m_n} (T e_{ni}, e_{ni}).$$
