Assume $(X, \mathcal{B}, \mu)$, $(Y, \mathcal{C}, \nu)$ are σ-finite measure spaces. Given a measurable function $k : X \times Y \to \mathbb{C}$, the integral operator T_k with kernel k is defined by

$$(T_k f)(x) = \int_Y k(x,y) f(y) \, d\nu$$

for measurable $f : Y \to \mathbb{C}$, whenever the formula makes sense.

Theorem

Assume $k \in L^2(X \times Y, d(\mu \times \nu))$. Then $T_k : L^2(Y, d\nu) \to L^2(X, d\mu)$ is a well-defined Hilbert-Schmidt operator and $\|T_k\|_2 \leq \|k\|_2$. Moreover, the map $k \mapsto T_k$ defines a unitary isomorphism $L^2(X \times Y, d(\mu \times \nu)) \simeq$

$$\simeq HS(L^2(Y, d\nu), L^2(X, d\mu)).$$

Proof

For $f \in L^2(Y, d\nu)$, we have

$$\left| \int_Y k(x,y) f(y) \, d\nu \right|^2 \leq \left(\int_Y |k(x,y)|^2 \, d\nu(y) \right) \left(\int_Y |f(y)|^2 \, d\nu(y) \right)$$

$$= \int_Y |k(x,y)|^2 \, d\nu(y) \, \|f\|_2^2.$$

Since $k \in L^2(X \times Y, d(\mu \times \nu))$, we have $\int_Y |k(x,y)|^2 \, d\nu(y) < \infty$ for μ-a.e. x by Tonelli.

Therefore $(T_K f)(x)$ is well-defined for μ-a.e. x.
The function $x \mapsto (T_K f)(x)$ is measurable
By the same arguments as in the proof of
the Fubini-Tonelli theorem (exercise?). Then

$$\|T_K f\|_2^2 = \int_X |(T_K f)(x)|^2 d\mu$$

$$\leq \int_{X\times Y} |k(x,y)|^2 d(\mu\times\nu)(x,y) \, \|f\|_2^2$$

$$= \|k\|_2^2 \, \|f\|_2^2 .$$

Therefore T_K is bounded, with $\|T_K\| \leq \|k\|_2$.

For $f \in L^2(X, d\mu)$, $g \in L^2(Y, d\nu)$, define
$k_{f,g}(x,y) \stackrel{\text{def}}{=} f(x)\overline{g(y)}$. Then
$T_K = \Theta_{f,g} = (\cdot, g) f$. We also have

$$(k_{f_1,g_1}, k_{f_2,g_2}) = (f_1, f_2)(g_2, g_1),$$

$$(\Theta_{f_1,g_1}, \Theta_{f_2,g_2})_{HS} = \mathrm{Tr}(\Theta^*_{f_2,g_2} \Theta_{f_1,g_1})$$

$$= \mathrm{Tr}(\Theta_{g_2,f_2}\Theta_{f_1,g_1}) = (f_1, f_2)\,\mathrm{Tr}(\Theta_{g_2,g_1})$$

$$= (f_1, f_2)(g_2, g_1).$$

Therefore $K \mapsto T_K$ is an isometric map

$$\mathrm{span}\{k_{f,g}\} \to F(L^2(Y, d\nu), L^2(X, d\mu)),$$

where the spaces are considered with

the L^2-norm and the Hilbert-Schmidt norm, resp. It follows that this map extends by continuity to a unitary

$$U : L^2(X\times Y, d(\mu\times\nu)) \to HS(L^2(Y,d\nu), L^2(X,d\mu)).$$

It remains to show that $UK = T_K$ for all K, while for now we know this only for $K \in \operatorname{span}\{K_{f,g}\}$.

Take $K \in L^2(X\times Y, d(\mu\times\nu))$. Choose $K_n \in \operatorname{span}\{K_{f,g}\}$ s.t. $\|K-K_n\|_2 \xrightarrow[n\to\infty]{} 0$. Then $\|UK - UK_n\|_2 = \|UK - T_{K_n}\|_2 \xrightarrow[n\to\infty]{} 0$. In particular, $\|UK - T_{K_n}\| \xrightarrow[n\to\infty]{} 0$. On the other hand, $\|T_K - T_{K_n}\| = \|T_{K-K_n}\| \le \|K-K_n\|_2 \xrightarrow[n\to\infty]{} 0$. Hence $UK = T_K$. ∎

Corollary

If $K_1, K_2 \in L^2(X\times Y, d(\mu\times\nu))$, then

$$\operatorname{Tr}(T_{K_2}^* T_{K_1}) = (K_1, K_2).$$

Consider now one σ-finite measure space $(X, \mathcal{B}, \mu)$. Since trace-class operators are Hilbert-Schmidt, any trace-class operator on $L^2(X, d\mu)$ has the form T_K for a unique

$K \in L^2(X \times X, d(\mu \times \mu))$. There is <u>still no complete characterization</u> of kernels K s.t. T_K is trace-class.

Observe that if $K_1, K_2 \in L^2(X \times X, d(\mu \times \mu))$, then $T_{K_1} T_{K_2} = T_{K_1 * K_2}$, where

$$(K_1 * K_2)(x_1, x_2) \overset{\text{def}}{=} \int_X K_1(x_1, x) K_2(x, x_2) \, d\mu(x).$$

Also, $T_K^* = T_{K^*}$, where $K^*(x,y) \overset{\text{def}}{=} \overline{K(y,x)}$.

It follows that if $K_1, K_2 \in L^2(X \times X, d(\mu \times \mu))$, then

$$\operatorname{Tr}(T_{K_1 * K_2}) = \operatorname{Tr}(T_{K_1} T_{K_2}) = \operatorname{Tr}(T_{K_1} T_{K_2}^*)$$

$$= (K_1, K_2^*) = \int_{X \times X} K_1(x,y) K_2(y,x) \, d(\mu \times \mu)(x,y)$$

$$= \int_X (K_1 * K_2)(x,x) \, d\mu(x).$$

Informally, this shows that if T_K is of trace class, then

$$\operatorname{Tr} T_K = \int_X K(x,x) \, d\mu(x),$$

but one has to be careful: the set $\Delta(X) \overset{\text{def}}{=} \{(x,x) : x \in X\} \subset X \times X$ often has $(\mu \times \mu)$-measure zero, so $K|_{\Delta(X)}$ is not well-defined by T_K.

Theorem (Duflo, Brislawn)

Assume X is a separable metric space, μ is a σ-finite Borel measure on X, Assume $K: X \times X \to \mathbb{C}$ is a Borel function s.t. T_K is of trace class and K is continuous at (x,x) for μ-a.e. $x \in X$.

Then

$$\operatorname{Tr} T_K = \int_X K(x,x)\, d\mu(x).$$

Proof

We will only prove the theorem under the following stronger assumptions:

- X is compact;
- μ is finite;
- K is continuous on $X \times X$.

Choose a partition

$$X = A_{n,1} \sqcup \dots \sqcup A_{n,m_n}$$

of X into ~~finitely~~ many sets of diameter $< \frac{1}{n}$ each (namely, cover X by finitely many balls $B_1, \dots, B_{m_n}$ of radius $\frac{1}{2n}$, then let $A_{nk} = B_k \setminus \bigcup_{\ell=1}^{k-1} B_\ell$).

Let $H_n \stackrel{\text{def}}{=} \operatorname{span}\{\mathbb{1}_{A_{nk}} : 1 \le k \le m_n\} \subset L^2(X, d\mu)$, and let $P_n : L^2(X, d\mu) \to H_n$ be the orthogonal projection

Then $P_n \to I$ strongly. Indeed, since $C(X)$ is dense in $L^2(X, d\mu)$, it suffices to show that $\|f - P_n f\|_2 \to 0$ for all $f \in C(X)$.

Take $f \in C(X)$. Fix $\varepsilon > 0$. We can find n_0 s.t. if $d(x_1, x_2) < \frac{1}{n_0}$, then $|f(x_1) - f(x_2)| < \varepsilon$.

Then, for all $n \geq n_0$,

$$\|f - P_n f\|_2^2 = \left\| f - \sum_{k=1}^{m_n} \frac{1}{\mu(A_{nk})} (f, \mathbb{1}_{A_{nk}}) \mathbb{1}_{A_{nk}} \right\|_2^2$$

$$= \left\| \sum_{k=1}^{m_n} \left(f - \frac{1}{\mu(A_{nk})} \int_{A_{nk}} f \, d\mu \right) \mathbb{1}_{A_{nk}} \right\|_2^2,$$

where $\frac{1}{\mu(A_{nk})} \int_{A_n} f \, d\mu$ is interpreted as 0 if $\mu(A_{nk}) = 0$. Therefore

$$\|f - P_n f\|_2^2 = \sum_{k=1}^{m_n} \int_{A_{nk}} \left| f(x) - \frac{1}{\mu(A_{nk})} \int_{A_{nk}} f \, d\mu \right|^2 d\mu$$

Note that if $x \in A_{nk}$, then for any $x' \in A_{nk}$ we have $|f(x) - f(x')| < \varepsilon$, hence $\left| f(x) - \frac{1}{\mu(A_{nk})} \int_{A_{nk}} f(x') \, d\mu(x') \right| \leq \varepsilon$ if $\mu(A_{nk}) > 0$.

Thus,

$$\|f - P_n f\|_2^2 \leq \varepsilon^2 \sum_{k=1}^{m_n} \mu(A_{nk}) = \varepsilon^2 \mu(X).$$

This proves our claim.

Then $T_f T_K = \lim_{n \to \infty} T_f (P_n T_K P_n)$

$$= \lim_{n\to\infty} \sum_{k=1}^{m_n} \frac{1}{\mu(A_{nk})} (T_K \mathbb{1}_{A_{nk}}, \mathbb{1}_{A_{nk}})$$

$$= \lim_{n\to\infty} \sum_{k=1}^{m_n} \mu(A_{nk}) \frac{1}{\mu(A_{nk})^2} \int_{A_{nk}\times A_{nk}} K(x,y)\, d(\mu\times\mu)(x,y),$$

where again we interpret a summand as 0 if it corresponds to $\mu(A_{nk})=0$.

Similarly to our proof of $\|f - P_n f\|_2 \to 0$, for any $\varepsilon > 0$ we can find n_0 s.t. for all $n \geq n_0$ we have

$$\left| \frac{1}{\mu(A_{nk})^2} \int_{A_{nk}\times A_{nk}} K(x,y)\, d\mu(x,y) - \frac{1}{\mu(A_{nk})} \int_{A_{nk}} K(x,x)\, d\mu(x) \right| < \varepsilon$$

for all $k = 1, \dots, m_n$ s.t. $\mu(A_{nk}) > 0$.

It follows that

$$\sum_{k=1}^{m_n} \mu(A_{nk}) \frac{1}{\mu(A_{nk})^2} \int_{A_{nk}\times A_{nk}} K(x,y)\, d(\mu\times\mu)(x,y)$$

becomes arbitrarily close to

$$\sum_{k=1}^{m_n} \mu(A_{nk}) \frac{1}{\mu(A_{nk})} \int_{A_{nk}} K(x,x)\, d\mu(x) = \int_X K(x,x)\, d\mu(x).$$

Thus,

$$\operatorname{Tr} T_K = \int_X K(x,x)\, d\mu(x).$$

∎