

Compulsory assignment in MAT4410, Fall 2026

Throughout the assignment we assume that H is a (nonzero) Hilbert space and T is a bounded linear operator on H .

The *numerical range* of T is defined by

$$W(T) := \{(Tx, x) : \|x\| = 1\} \subset \mathbb{C}.$$

The *numerical radius* of T is defined by

$$w(T) := \sup\{|\lambda| : \lambda \in W(T)\}.$$

Problem 1.

(a) Show that if numbers $\lambda, \mu \in \mathbb{C}$ satisfy $\lambda \notin \sigma(T)$ and $|\mu - \lambda| < \|(T - \lambda 1)^{-1}\|^{-1}$, then $\mu \notin \sigma(T)$. Conclude that if $(\lambda_n)_n$ is a sequence in $\mathbb{C} \setminus \sigma(T)$ converging to an element of $\sigma(T)$, then $\|(T - \lambda_n 1)^{-1}\| \rightarrow \infty$.

Hint: Neumann series.

(b) Show using (a) that if $\lambda \in \partial\sigma(T)$, then there exists a sequence of unit vectors x_n such that $\|Tx_n - \lambda x_n\| \rightarrow 0$. Conclude that $\lambda \in \overline{W(T)}$.

Here $\partial\sigma(T)$ denotes the boundary of $\sigma(T)$, that is, the set of points $\lambda \in \mathbb{C}$ such that every neighborhood of λ intersects both $\sigma(T)$ and $\mathbb{C} \setminus \sigma(T)$.

(c) Assume that T is normal, that is, $TT^* = T^*T$. Show that $w(T) = \|T\|$.

Hint: recall that $r(T) = \|T\|$.

Problem 2. Assume T is normal.

(a) Show that if $Tx = \lambda x$, then $T^*x = \bar{\lambda}x$. More generally, show that if $\|Tx_n - \lambda x_n\| \rightarrow 0$ for some x_n , then $\|T^*x_n - \bar{\lambda}x_n\| \rightarrow 0$.

(b) Show that if x and y are eigenvectors of T with different eigenvalues, then $x \perp y$. More generally, show that if $(x_n)_n$ and $(y_n)_n$ are sequences of unit vectors such that $\|Tx_n - \lambda x_n\| \rightarrow 0$ and $\|Ty_n - \mu y_n\| \rightarrow 0$ for some $\lambda \neq \mu$, then $(x_n, y_n) \rightarrow 0$.

(c) Strengthen 1(b) by showing that $\overline{W(T)}$ contains the convex hull of $\partial\sigma(T)$.

Problem 3.

(a) Show that if $x_0, \dots, x_{n+1}$ are $n+2$ points in $\mathbb{R}^n$, then any point in their convex hull is a convex combination of $n+1$ of these points.

Hint: to modify coefficients in a convex combination of $x_0, \dots, x_{n+1}$ observe that these points must be *affinely dependent*, meaning that there exist numbers $c_0, \dots, c_{n+1} \in \mathbb{R}$, not all of which are zero, such that $\sum_{i=0}^{n+1} c_i = 0$ and $\sum_{i=0}^{n+1} c_i x_i = 0$.

(b) Prove the following result due to Carathéodory: if $K \subset \mathbb{R}^n$ is a compact set, then its convex hull $\text{conv } K$ is closed and equal to

$$\left\{ \sum_{i=0}^n c_i x_i : x_i \in K, c_i \geq 0, \sum_{i=0}^n c_i = 1 \right\}.$$

(c) Assuming again that $K \subset \mathbb{R}^n$ is compact, show that $\text{conv } K$ coincides with the intersection of closed balls containing K .

(d) Show also that if $K \subset \mathbb{R}^n$ is compact, then every point $x \in K \setminus \partial K$ is a convex combination of two boundary points of K . Conclude that

$$\text{conv } K = \text{conv } \partial K.$$

Problem 4. Combine the previous results to prove that if T is normal, then

$$\overline{W(T)} = \text{conv } \sigma(T).$$

Hint: to prove the inclusion $\subset$ observe that if $\sigma(T) \subset \overline{B_d(\lambda)}$, then $\|T - \lambda 1\| \leq d$ and hence $w(T - \lambda 1) \leq d$ by 1(c).

(The result is not true for general bounded operators. One can nevertheless show that $W(T)$ is always convex, this is the Toeplitz–Hausdorff theorem, and then we can at least conclude that $\text{conv } \sigma(T) \subset \overline{W(T)}$.)