

1. Assume X is a metric space, $A \subset X$ is a nowhere dense subset and $U \subset X$ is an open subset. Take the closure $Y = \bar{U} \subset X$. Show that $A \cap Y$ is nowhere dense in Y .

2. Recall that a subset of a metric space is called a G_δ -set if it is the intersection of countably many open sets. Show that a subset of a metric space X is residual if and only if it contains the intersection of countably many dense open sets. Conclude that if X is complete, then a subset of X is residual if and only if it contains a dense G_δ -subset of X .

3. Consider the function space $C[a, b]$ with the supremum-norm. Show that the subset of functions $f \in C[a, b]$ that are not differentiable at any point is residual (in particular, dense) in $C[a, b]$.

Hint: observe that the complement of this set is contained in $\cup_n D_n$, where

$$D_n = \{f : |f(x) - f(x_0)| \leq n|x - x_0| \text{ for some } x_0 \text{ and all } x\}.$$

4. Assume X and Y are complete metric spaces and $f: X \times Y \rightarrow \mathbb{C}$ is a function that is continuous in each variable. Show that the set of points of continuity of f is residual (in particular, dense) in $X \times Y$.

Hint: For $\varepsilon > 0$, consider the set U_ε of points (x, y) such that $|f(x_1, y_1) - f(x_2, y_2)| < \varepsilon$ for all (x_1, y_1) and (x_2, y_2) sufficiently close to (x, y) . Observe that U_ε is open and the set of points of continuity of f is $\cap_n U_{1/n}$. It follows that it suffices to show that the sets U_ε are dense in $X \times Y$. For this, in turn, consider the sets $A_n \subset X \times Y$ consisting of points (x, y) such that $|f(x, y) - f(x', y)| < \varepsilon/4$ whenever $d(x, x') < 1/n$, and observe that by assumption $\cup_n A_n = X \times Y$.