

1. Assume X and Y are normed spaces.

(1) Show that any two norms on the vector space $X \oplus Y$ such that the maps $X \mapsto X \oplus Y$, $x \mapsto (x, 0)$, and $X \oplus Y \rightarrow X$, $(x, y) \mapsto x$, are continuous and similar properties hold for Y , are equivalent.

(2) For $1 \leq p < \infty$, show that the formula

$$\|(x, y)\|_p = (\|x\|^p + \|y\|^p)^{1/p}$$

defines a norm on $X \oplus Y$ satisfying the conditions in (1).

2. Assume we are given a normed space X and a closed subspace $Y \subset X$. The *annihilator* of Y in X^* is defined by

$$Y^\perp = \{f \in X^* \mid f = 0 \text{ on } Y\}.$$

(1) Explain how elements of $Y^\perp$ can be considered as linear functionals on X/Y and elements of $X^*/Y^\perp$ can be considered as linear functionals on Y .

(2) Show that (1) gives rise to isometric(=preserving the norms) isomorphisms

$$(X/Y)^* \cong Y^\perp \text{ and } Y^* \cong X^*/Y^\perp.$$

3. Follow the proof of the uniform boundedness principle to conclude the following: if X and Y are normed spaces and $(T_i: X \rightarrow Y)_{i \in I}$ is a collection of bounded linear operators such that $\sup_i \|T_i\| = +\infty$, then the set

$$\{x \in X : \sup_i \|T_i x\| < \infty\}$$

is meager in X .

4. Assume $X_1, \dots, X_n$ ($n \geq 2$) are normed spaces, all of which except possibly one are complete, Y is another normed space and $B: X_1 \times \dots \times X_n \rightarrow Y$ is an n -linear map, meaning that it is linear in each variable. Show by induction that if B is continuous in each variable, then there exists $C \geq 0$ such that

$$\|B(x_1, \dots, x_n)\| \leq C\|x_1\| \dots \|x_n\|$$

for all $x_1 \in X_1, \dots, x_n \in X_n$. Conclude that B is continuous. (The case $n = 2$ was done in the class.)

5. Assume X and Y are normed spaces and $T: X \rightarrow Y$ is a linear operator such that $f \circ T \in X^*$ for every $f \in Y^*$. Show that T is bounded.