

1. Consider a σ -finite measure space $(\Omega, \mathcal{B}, \mu)$, an essentially bounded measurable function f and $1 \leq p \leq \infty$. Consider the operator A on $L^p(\Omega, d\mu)$ of multiplication by f , so

$$Ag = fg.$$

(1) Show that $\|A\| = \|f\|_\infty$.

(2) Show that the point spectrum $\sigma_p(A)$ consists of all numbers $\lambda \in \mathbb{C}$ such that $\mu(f^{-1}(\lambda)) > 0$.

(3) Show that the spectrum $\sigma(A)$ consists of all numbers $\lambda \in \mathbb{C}$ such that for every open neighbourhood U of λ we have $\mu(f^{-1}(U)) > 0$. This is called the *essential range* of f and can alternatively be described as the support of the Borel measure $f_*\mu$.

(4) The name “essential range” is a bit unfortunate, since it is in general impossible to modify f on a set of measure zero so that the essential range becomes equal $f(\Omega)$ (think of a countable Ω and an uncountable essential range). Show that the essential range coincides with the closure of the set of points $\lambda \in f(\Omega)$ such that for every open neighbourhood U of λ we have $\mu(f^{-1}(U)) > 0$.

2. (1) Follow the proof of the equality $\sigma(T) = \sigma(T^t)$ given in the lectures to conclude the following. Given a complex Banach space X and $T \in L(X)$, we have $\lambda \in \sigma(T)$ if and only if one of the following conditions is satisfied:

- (i) there is a sequence of unit(=norm one) vectors $x_n \in X$ such that $\|Tx_n - \lambda x_n\| \rightarrow 0$;
- (ii) there is $f \in X^* \setminus \{0\}$ such that $T^t f = \lambda f$.

(2) In a similar way, show that if H is a Hilbert space and $A \in L(H)$, then $\lambda \in \sigma(A)$ if and only if one of the following conditions is satisfied:

- (i) there is a sequence of unit vectors $x_n \in H$ such that $\|Ax_n - \lambda x_n\| \rightarrow 0$;
- (ii) there is $x \in H \setminus \{0\}$ such that $A^*x = \bar{\lambda}x$.

(3) A bounded operator A on a Hilbert space H is called *self-adjoint* if $A^* = A$. Conclude from (2) that for any self-adjoint operator A we have $\sigma(A) \subset \mathbb{R}$, and $\lambda \in \sigma(A)$ if and only if there is a sequence of unit vectors $x_n \in H$ such that $\|Ax_n - \lambda x_n\| \rightarrow 0$.

Therefore, even though $\sigma(A)$ does not always consist of only eigenvalues, its elements are the real numbers that are “almost eigenvalues”.