

1. (1) A sequence of real numbers $(a_n)_{n=1}^{\infty}$ is called *subadditive* if $a_{m+n} \leq a_m + a_n$ for all $m, n \geq 1$. Show that for any such sequence the limit

$$\lim_{n \rightarrow \infty} \frac{a_n}{n} \in [-\infty, +\infty)$$

exists and coincides with $\inf_n \frac{a_n}{n}$.

(2) A sequence of nonnegative numbers $(a_n)_{n=1}^{\infty}$ is called *submultiplicative* if $a_{m+n} \leq a_m a_n$ for all $m, n \geq 1$. Show that for any such sequence the limit

$$\lim_{n \rightarrow \infty} a_n^{1/n} \in [0, +\infty)$$

exists and coincides with $\inf_n a_n^{1/n}$. This shows, in particular, that for any bounded linear operator T on a normed space the limit $\lim_n \|T^n\|^{1/n}$ exists without relying on the spectral radius formula.

2. Assume X and Y are normed spaces and consider a bounded linear operator $T: X \rightarrow Y$ of finite rank, so that $T = \sum_{k=1}^n f_k(\cdot) y_k$ for some $y_1, \dots, y_n \in Y$ and $f_1, \dots, f_n \in X^*$. Show that for the transpose $T^t: Y^* \rightarrow X^*$ of T we have

$$T^t = \sum_{k=1}^n y_k(\cdot) f_k,$$

where we view $y_k \in Y$ as elements of Y^{**} . In particular, T^t is again of finite rank.

3. (1) Assume K is a metric space with the following property: for every $\varepsilon > 0$, there is a precompact subset $K_\varepsilon \subset K$ such that $d(x, K_\varepsilon) < \varepsilon$ for all $x \in K$. Show that K is precompact.

(2) Prove the following theorem due to Mazur. Assume X is a normed space and $K \subset X$ is a precompact subset. Then its convex hull is precompact as well. By the convex hull one means the smallest convex set containing K , or more concretely, the set of all vectors of the form $\lambda_1 x_1 + \dots + \lambda_n x_n$, where $n \geq 1$, $x_1, \dots, x_n \in K$, $\lambda_i \in [0, 1]$ and $\lambda_1 + \dots + \lambda_n = 1$.

Hint: consider first the case of a finite set.